

MATHEMATICS teacher

SEPTEMBER 2013

levels of infinity



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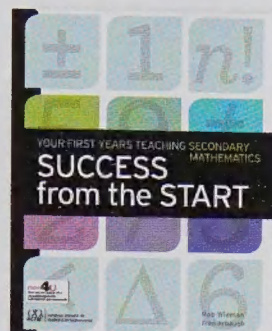
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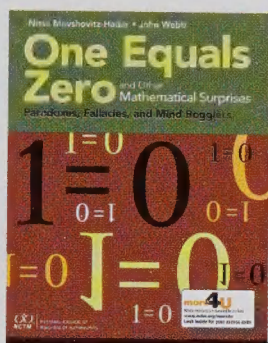
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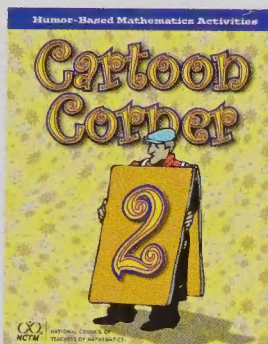


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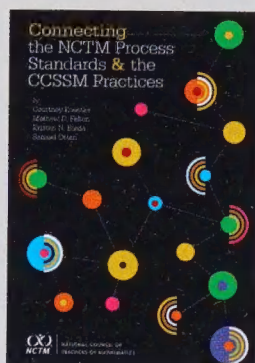
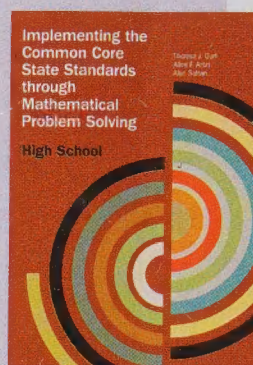
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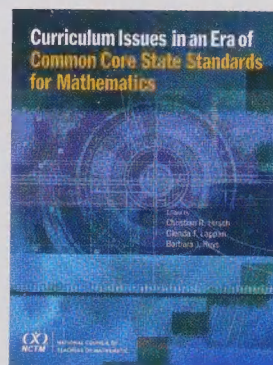
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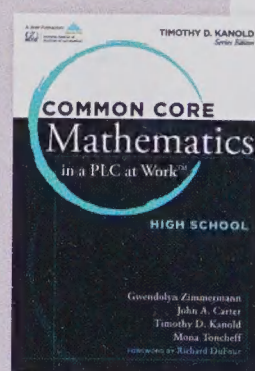
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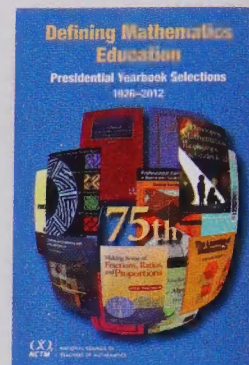


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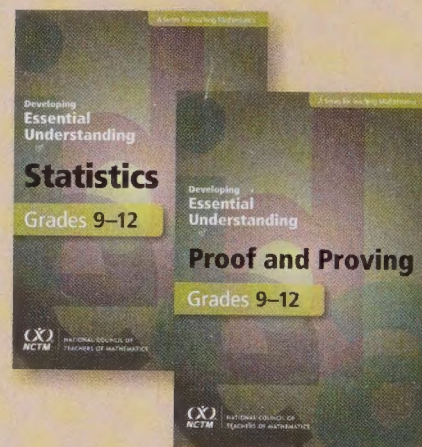
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on the cover

"Oh, yeah? Well, mine is infinity plus one!" As this snippet of schoolyard dialogue indicates, students are typically fascinated by the idea of infinity, misunderstanding it as so many do. Teo J. Paoletti tries to rectify that situation with his article "Are All Infinities Created Equal?" (pp. 98–103). Blending the history of mathematics with lots of reasoning and sense making, Paoletti presents some accessible proofs regarding cardinality and one-to-one correspondence that are sure to increase your students' understanding of infinity.

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- Gordon (p. 104): A fuller explanation of the Curriculum Foundations Project plus additional examples and ideas for implementation
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in the next issue

Coming in October 2013:

- "Spaghetti Sine Curves: Virtual Environments for Reasoning and Sense Making," by S. Asli Özgün-Koca, Michael Todd Edwards, and Michael Meagher
- "Reinforcing Geometric Properties with Shapedoku Puzzles," by Jeffrey J. Wanko and Jennifer V. Nickell
- "Open-ended Questions and the Process Standards," by Wendy B. Sanchez

ARC LENGTH REVISITED

James Perna's note on arc length (Reader Reflections, *MT* Dec. 2012/Jan. 2013, vol. 106, no. 5, pp. 328–29) is a thoroughgoing discussion of a natural and valuable question. Last year I worked out construction of tractable arc length problems, which may be useful. The methods produce doable integrals from manipulation of arbitrary functions.

For curves in two dimensions, choose two convenient functions $a(t)$ and $b(t)$. Then

$$(x(t), y(t)) = \left(\int (a^2 - b^2) dt, \int 2ab dt \right)$$

gives parametric equations and a solvable integral for the arc length.

In the May 2007 *College Mathematics Journal*, reprinted in *The Calculus Collection: A Resource for AP and Beyond* (Mathematical Association of America 2010), John Ferdinands developed the following algorithm for curves in the form $y = f(x)$: Choose $g(x)$. Then

$$f(x) = \frac{1}{2} \int_a^b \left(g(x) - \frac{1}{g(x)} \right) dx$$

will have arc length given by

$$L = \frac{1}{2} \int_a^b \left(g(x) + \frac{1}{g(x)} \right) dx.$$

Gary Miller

gmiller@pi.ac.ae

The Petroleum Institute

Abu Dhabi, UAE, Dec. 12, 2012

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged. Letters to be considered for publication should be in MS Word document format and sent to mt@nctm.org. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

LOGGING HARMONIC CONVERGENCE

Kudos to Matson and Grigoriadou ("Don't Fret over Exponential Functions," *MT* Dec. 2012/Jan. 2013, vol. 106, no. 5) for locating fret positions of stringed instruments as an application of exponential functions. I suggest the "no fretting" follow-up of finding *harmonic notes*—ghostly sustaining tones that are prepared by the "fretting hand" not by pressing a string down against the fingerboard but by simply lightly touching the string at a point that divides its length into an $(m - n) : n$ ratio, in which m and n are (small) natural numbers (with $n = 1, 2, 3, \dots, m - 1$) and the fraction $(m - n) : n$ is in simplest form. The distance of this point from the guitar's bridge is $L \cdot (m - n)/m$, where L is the full length of the open (unfretted) string. Although such points can be located one by one with tape measure and fraction arithmetic, already existing landmarks on a guitar (namely, the metal bars called frets) would be convenient to use.

The authors' exploration lets us state that the distance from the bridge of fret x (fret numbering starts at the nut) is Lr^x , where r is the value $1/2^{1/12}$ that they discussed. Equating our two "distance from the bridge" expressions yields $(m - n)/m = r^x$. Applying logarithms yields

$$x = \frac{\log\left(\frac{m-n}{m}\right)}{\log r}.$$

Note that $m = 2, 3, 4$ produces values of x that are exactly (or nearly exactly) a whole number (see **table 1** [Lesser], and these are used in hits such as "Bar-racuda" (by Heart) and "Roundabout" (by Yes). A non-whole-number fret

position can still be played efficiently by placing one's finger appropriately between its two surrounding frets.

Lawrence M. Lesser

Lesser@utep.edu

The University of Texas at El Paso

Feb. 19, 2013

APPLYING VISIBLE THINKING

I read "Visible Thinking in High School Mathematics," by Emily Sliman (*MT* March 2013, vol. 106, no. 7, pp. 502–7) a day before I introduced imaginary numbers in my second-year algebra class. I was easily able to set up three stations with questions that linked topics on the recent chapter on quadratic functions and the upcoming unit:

- "What do you know about roots of a function?"
- "What do you know about square roots?"
- "What do you know about imaginary numbers?"

The students enjoyed the lesson. Even those who initially wrote, "I don't know," looked back at what others had jotted down and attempted to join the conversation. On the last rotation, one student wrote, "I feel like this is related to the other questions and has to do with what we're learning next." At the end, she was the one to ask whether we could repeat the activity at the end of the unit to compare posters. It was a great idea, one I would recommend to any teacher. The only difficulty I faced was getting students to stop writing irrelevant notes to one another!

Jaime Marts

jmemarts@hotmail.com

Marylawn of the Oranges Academy

South Orange, NJ, Mar. 17, 2013

Harmonic m	2	3	3	4	5	6
Node n	1	1	2	1	1	1
Fret x	12	7.02	19.02	4.98	3.86	3.16

Table 1 (Lesser)

02	15	23	06	19	02	15	23	06
08	16	04	12	25	08	16	04	12
14	22	10	18	01	14	22	10	18
20	03	11	24	07	20	03	11	24
21	09	17	05	13	21	09	17	05
02	15	23	06	19	02	15	23	06
08	16	04	12	25	08	16	04	12
14	22	10	18	01	14	22	10	18
20	03	11	24	07	20	03	11	24

Fig. 1 (Becker)

MORE ON MAGIC SQUARES

In Lawrence Lesser's reflection "The Center of a Magic Square" (*MT* Mar. 2013, vol. 106, no. 7, p. 486), the value in the center square of any odd-order magic square is said to be the mean. This is incorrect.

Consider the 9×9 array in **figure 1 (Becker)**. Choose any 5×5 square in the array. There are 25 ways to do so. The upper-left corner of the array (shaded in the figure) has center square 10. The lower-right corner of the array (also shaded) has center square 16. Each of these fifth-order magic squares has the following properties: Rows, columns, main diagonals, and broken diagonals sum to 65; and each of these 25 fifth-order panmagic squares has a different natural number from 1 to 25 in its center square. In only one of these odd-order magic squares is the mean in the center square.

Robert H. Becker

rbecker@writeme.com

*Governor Mifflin Senior High School (retired)
Shillington, PA, Mar. 14, 2013*

REDISTRICTING APPLICATION

Regarding "Decennial Redistricting: Rich Mathematics in Context" (*MT* Oct. 2012, vol. 106, no. 3, pp. 206–11), see the diagram shown in **figure 1 (Cibes)**. It is virtually identical to one in *The Power of Numbers: A Teacher's Guide to Mathematics in a Social Studies Context* (http://www.esrnational.org/store/index.php?main_page=product_

[info&cPath=8&products_id=33](http://www.esrnational.org/store/index.php?main_page=product_info&cPath=8&products_id=33)), a workbook published by the Citizens for Social Responsibility (Boston) and aimed at middle school teachers. The discussion below is based on the original activity, which I have revised somewhat for my own use in my first-year college classes on equity and decision making.

The figure shows a mythical state that has been apportioned 10 seats in the U.S. House of Representatives and is required to create 10 districts of equal populations. Each *R*, *D*, and *U* represents 5,000 people considered Republican, Democrat, or Unaffiliated, respectively, and the total population is 350,000.

Note that apportionment is based on total population, including children and other unregistered voters, so these two nonvoting groups have been arbitrarily assigned to the *R*, *D*, or *U* groups. In real life, we would have additional information, at the very least about the geographic distribution of registered voters and their affiliations and whether unaffiliated voters in a state tend to prefer one party over another, despite not affiliating themselves with a particular party.

The rules are simple:

- Every town (shaded region) with 35,000 or more people must lie in a single district.
- Every part of each district must be accessible from another part without crossing district boundaries.

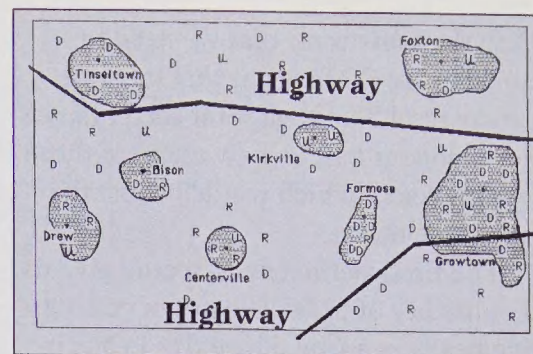


Fig. 1 (Cibes)

I offered students three scenarios: (1) create districts according to the best interests of the Democratic Party; (2) create districts according to the best interests of the Republican Party; or (3) create "neutral" districts. In any scenario, students did not assume any preference by persons considered unaffiliated. Students then wrote reports describing their reasoning and justifying how their goals had been met by their districting. Students also contrasted this mythical process with what they imagined—or knew—a real-life process to be.

Margaret Cibes

cibesm@comcast.net

*University of Hartford (retired)
Hartford, CT, June 10, 2013*

PROBLEMS 22 AND 23, MAY 2012 CALENDAR

Problem 22

Problem 22 in the May 2012 Calendar (*MT* Apr. 2012, vol. 105, no. 8, p. 601) is stated as follows:

A fair coin is tossed five times. What is the probability of obtaining at least three consecutive heads?

One alternate solution would be to look at the problem as a truth table, with 32 rows and *T* standing for heads and *F* standing for tails (see **table 1 [Forringer]**). (We could, of course, just use *H* and *T*.) Notice that the rows with the asterisk (*) meet the condition of at least three consecutive heads.

A second alternate solution would involve using the binomial expansion (in this case, *H* stands for heads, and *T* stands for tails):

$$(H + T)^5 = H^5 + 5H^4T + 10H^3T^2 + 10H^2T^3 + 5HT^4 + T^5$$

Now the only terms that we need to consider are those terms that have an exponent of 3 or greater for the H (since the exponent tells us how many of those letters occur), which would be just the first three terms.

The first coefficient, 1, would give us the number of possible ways of getting five heads in a row. Since five heads in a row satisfies the condition of the problem, we have one success.

The second coefficient, 5, would give us the number of possible ways of getting 4 heads. Four of those 5 would be

3 in a row, so that gives us a total of 5 ways with the first two terms. (HHTHH would be the only arrangement that would not include 3 Hs in a row.)

The third coefficient, 10, would give us the number of possible ways of getting 3 heads. The only ones that would meet the condition of “in a row” would be where the first three were heads, or the second three were heads, or the last three were heads.

So the first term gives us 1 way, the second term gives us 4 ways, and the third term gives us 3 ways, for a total

of 8 ways. Eight out of 32 (32 being the sum of all the coefficients of all the terms) gives us the correct answer of $1/4$.

Problem 23

Problem 23 in the May 2012 Calendar (*MT* Apr. 2012, vol. 105, no. 8, p. 601) reads:

In base 10, the number 31 represents a prime number. In base k , however, this two-digit number represents a perfect square. For what value or values of k , $3 < k < 10$, is 31 a perfect square?

The problem would be more interesting if we were to remove the restriction $3 < k < 10$. In fact, every perfect square that does not contain a factor of 3 can be expressed in the form 31_k . The solution for the original problem listed 5 and 8 as the base values that would yield a perfect square, but without the restriction the answer would have been

5, 8, 16, 21, 33, 40, 56, 65, 85, 96, 120, 133, 161, 176, . . .

Can readers or their students find the next three numbers in the series? Note that the difference between the first two elements of the set is 3; the difference between the third and fourth elements is 5; between the fifth and sixth is 7, and so on. The difference between the second and third elements is 8, between the fourth and fifth elements is 12; between the sixth and seventh elements is 16; between the eighth and ninth elements is 20; and so on.

Richard Forringer

dick.forringer@gmail.com

Durham Academy (retired)

Durham, NC, Oct. 15, 2012

PROBLEM 14, DECEMBER 2012 CALENDAR

Problem 14 of the December 2012 Calendar (*MT* Nov. 2012, vol. 106, no. 4, p. 281) was stated as follows:

Trapezoid $ABCD$ with bases AD and BC has diagonals meeting at E . The line through E parallel to the bases intersects the nonparallel sides at F and G . If $BC = 10$ and $AD = 15$, compute FG .

I would like to offer an alternate solution to this problem. Note that because

*	T	T	T	T	T
*	T	T	T	T	F
*	T	T	T	F	T
*	T	T	T	F	F
	T	T	F	T	T
	T	T	F	T	F
	T	T	F	F	T
	T	T	F	F	F
*	T	F	T	T	T
	T	F	T	T	F
	T	F	T	F	T
	T	F	T	F	F
	T	F	F	T	T
	T	F	F	T	F
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	T	F	F	F	F
*	F	T	T	T	T
*	F	T	T	T	F
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	F	T	F	T	T
	F	T	F	T	F
	F	T	F	F	T
	F	T	F	F	F
*	F	F	T	T	T
	F	F	T	T	F
	F	F	T	F	T
	F	F	T	F	F
	F	F	F	T	T
	F	F	F	T	F
	F	F	F	F	T
	F	F	F	F	F

Table 1 (Forringer)

triangle BCE is similar to triangle DAC , the altitudes of these triangles through point E are proportional; the scale ratio is 10:15, or 2:3.

An intriguing aspect of this problem is that the height of the trapezoid does not matter; the length FG , the parallel through the point of intersection of the diagonals, is fully determined by the length of the bases. Thus, we are free to make those two altitudes anything we like in the ratio of 2:3; I choose to make them 2 units and 3 units.

Now the height of the whole trapezoid is 5 units, and its area is easily calculated to be $5 \cdot (10 + 25)/2 = 62.5$ units². If we let $FG = x$, the area of the top trapezoid plus that of the bottom trapezoid must equal 62.5 units²—

$$\frac{2(10+x)}{2} + \frac{3(x+15)}{2} = 62.5$$

—which produces the correct solution, $x = 12$ units.

Ken McCaffrey

kenneth.mccaffrey@yahoo.com
Brattleboro Union High School (retired)
Brattleboro, VT, Oct. 21, 2012

PROBLEM 19, MARCH 2013 CALENDAR

Problem 19 in the March 2013 Calendar (*MT* Feb. 2013, vol. 106, no. 6, p. 441) is stated as follows:

The numerical value of the volume of a cylinder is twice the numerical value of its surface area. If the radius and the height are both integers, determine the smallest possible volume.

The volume of a cylinder is given by $V = \pi r^2 l$. The corresponding surface area is given by $A = 2\pi r^2 + 2\pi r l$. Letting $V = 2A$ yields $rl = 4(r + l) \rightarrow rl - 4r - 4l = 0 \rightarrow rl - 4r - 4l + 16 = 16$ or $(r - 4)(l - 4) = 16$. The five possible factorizations are shown in **table 1 (Goehl)**.

The smallest volume occurs for $r = 6$. This is the smallest volume for any r , integral or not. We can see this by taking the derivative of the volume with respect to r . Notice that $l = 16/(r - 4) + 4$, and so the volume is given by $\pi r^2 [16/(r - 4) + 4]$. Setting the derivative equal to 0 yields $r = 6$ and $V = 432\pi \approx 1357.168$.

$r - 4$	$l - 4$	r	l	V	A
16	1	20	5	2000π	1000π
8	2	12	6	864π	432π
4	4	8	8	512π	256π
2	8	6	12	432π	216π
1	16	5	20	500π	250π

Table 1 (Goehl)

$r - 4$	$l - 4$	r	l	V	A
36	1	42	7	12348π	4116π
18	2	24	8	4608π	1536π
12	3	18	9	2916π	972π
9	4	15	10	2250π	750π
6	6	12	12	1728π	576π
4	9	10	15	1500π	500π
3	12	9	18	1458π	486π
2	18	8	24	1536π	512π
1	36	7	42	2058π	686π

Table 2 (Goehl)

This procedure can be followed when V is any multiple, m , of A : $(r - 2m) \cdot (l - 2m) = 4m^2$. As an example, **table 2 (Goehl)** shows the results for $m = 3$.

For any m , the derivative set equal to 0 yields $r = 3m$, which corresponds to a smallest volume of $54m^3\pi$.

John F. Goehl Jr.

jgoehl@mail.barry.edu
Barry University
Miami Shores, FL, Feb. 26, 2013

HOW HIGH AM I?

Suppose that your aircraft’s altimeter is not working. Can you calculate your altitude by merely observing the horizon? Well, let’s investigate. Perhaps we’ll discover a few other facts while we’re at it.

We adopt the labeling notation shown in **figure 1 (Siegel)**, where h is our altitude, R is the radius of Earth, and θ is the declination of the horizon. Angle AOT has value θ because it is the complement of the complement of θ . The triangle reveals that $\cos \theta = R/(R + h)$, which is equivalent to $h = R(1 - \cos \theta)/\cos \theta$. Therefore, if we know Earth’s radius, then we know our

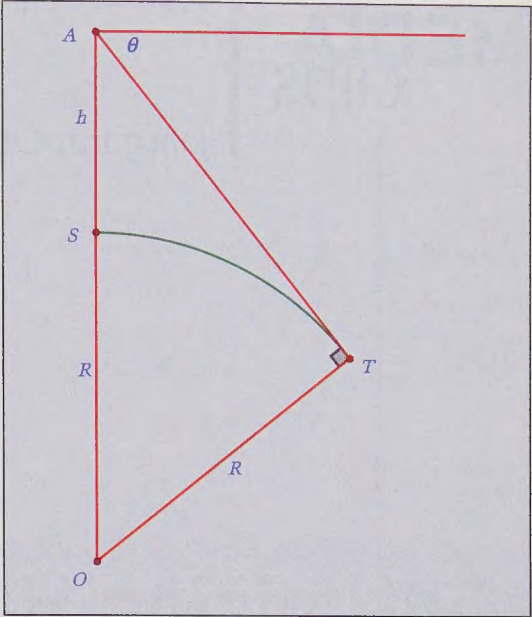


Figure 1 (Siegel)

altitude (assuming that we can accurately measure the angle of depression of the horizon). Conversely, if we know our altitude and the angle of declination to the horizon, then we can calculate Earth’s radius by solving our equation for R . That is, $R = h \cos \theta / (1 - \cos \theta)$.

We can also tell how much of Earth is in our purview. The triangle provides the information that the length of side AT is $R \tan \theta$, which is the distance from our craft to the horizon. On the other hand, the length of arc ST , the distance from the point on Earth directly below our craft to the horizon, is $R \theta$ (assuming that we measure θ in radians) since arc length along a circle equals the product of the radius and the central angle that intercepts the arc. Those two lengths are approximately equal if we are not too high. In that case, where θ is close to zero, θ and $\tan \theta$ are almost indistinguishable because $\tan \theta$ (the Maclaurin series for $\tan \theta$ begins with $\theta + \theta^3/3$) can be approximated by θ .

After concluding this investigation, I Googled “radius of the Earth,” whereupon I learned that a Persian mathematician, Abu al-Biruni (CE 973–1048), determined Earth’s radius in the same way (but from a mountain, not an aircraft). He did so nearly five hundred years before Columbus discovered the New World. It’s always a thrill to discover a link to the past.

Steven Siegel

siegel443@comcast.net
Niagara University (retired)
Niagara, NY, Mar. 20, 2012

Optimizing HDTV Viewing Angles

Your TV's screen size should depend on how far you'll be sitting from it, what you'll be watching, and your personal tastes and preferences. . . .

Viewing distance and picture quality: two parts of the screen size equation

Sitting the right distance from your TV is also an important part of optimizing the viewing experience. Because HDTVs offer so much more picture detail than old analog sets, you can sit quite a bit closer to them than you could with your old tube. In fact, sitting too far away may mean you miss some of the high-def detail you paid for. But you also want to avoid sitting too close to your TV. A good indication that you're sitting too close to a screen is if you find yourself distracted by the screen's "structure"—those rows and columns of pixels that look like tiny dots.

If you're not sure which screen size is best for your room size and viewing distance, check the guidelines [at right]. You can also use them to get an idea of appropriate viewing distance from your current TV. You'll notice that we provided a range for each screen size, rather than a hard and fast number. That's because the viewing distance you prefer will depend on your personal tastes, as well as what kind of material you'll most often watch. If you're going to watch lots of high-quality video—like DVDs and high-def programming—you can sit in the closer end of the range to see all the detail your HDTV can provide. On the other hand, if you'll still be watching lots of lower-quality sources, like analog cable, we recommend sitting at the higher end so that you'll notice less of the image's flaws.

Screen Diagonal (not for front projectors) (in in.)	Viewing Distance Range (in ft.)
26	3.25-5.50
32	4.00-6.66
37	4.63-7.71
40	5.00-8.33
46	5.75-9.50
52	6.50-10.80
58	7.25-12.00
65	8.13-13.50
70	8.75-14.75

Source: Julie Govan, "Choosing Screen Size and Placing Your TV," June 30, 2009, http://www.crutchfield.com/learn/learningcenter/home/TV_placement.html

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Edited by **Louis Lim**, louis.lim1@gmail.com
Thornhill Secondary School, Thornhill, ON,
Canada

Lionel Garrison, garrison@horacemann.org
Horace Mann School, Bronx, NY

Questions refer to an advertised 37-inch diagonal flat-screen HDTV. Although the dimensions of the screen display vary from one model HDTV to another, assume that the HDTV referred to in these questions has a screen display of 18.5 in. high by 32.5 in. wide.

Scenario 1

Suppose that this HDTV is mounted on a wall so that the bottom of the screen display is located at eye level (see **fig. 1** ["Optimizing HDTV Viewing Angles"]).

1. As the viewer moves closer to or farther away from the wall, the viewing angle (see **fig. 1** ["Optimizing HDTV Viewing Angles"]) will vary. Write a function that gives the viewing angle, $\theta(x)$ (measured in degrees) in terms of the viewer's distance from the wall x (in in.). Specify the domain of the function.
2. Crutchfield recommends a viewing distance between 4.63 ft. and 7.71 ft. Compute the viewing angle at the two given distances. Round your answers to the nearest tenth of a degree.

- Is there a distance from the wall that maximizes your viewing angle? Find this distance and the maximum viewing angle and interpret your results.

Scenario 2

Suppose, instead, that this HDTV is mounted on a wall so that the bottom of the screen display is located 10 in. above eye level (see **fig. 2** ["Optimizing HDTV Viewing Angles"]).

- Write a function that gives the viewing angle θ (in degrees) in terms of the viewer's distance from the wall x (in in.). Specify the domain of the function.
- Find the viewing angle for viewing distances of 4.63 ft. and 7.71 ft., rounding your answers to the nearest tenth of a degree. How do the viewing angles from the minimum and maximum recommended viewing distances in scenario 2 compare with those in scenario 1?

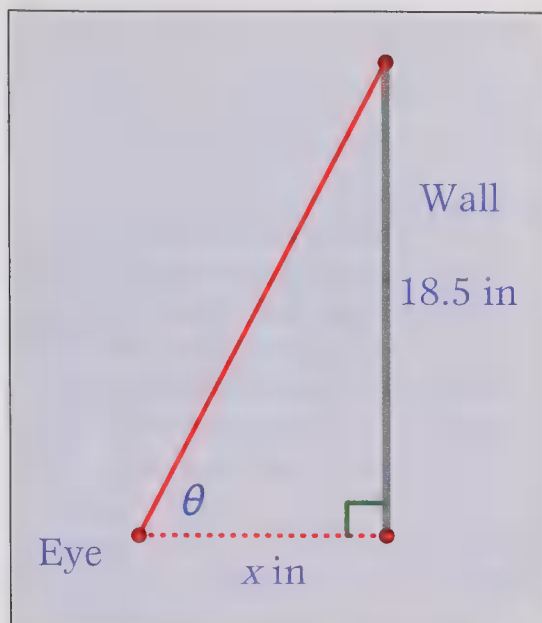


Fig. 1 ("Optimizing HDTV Viewing Angles")

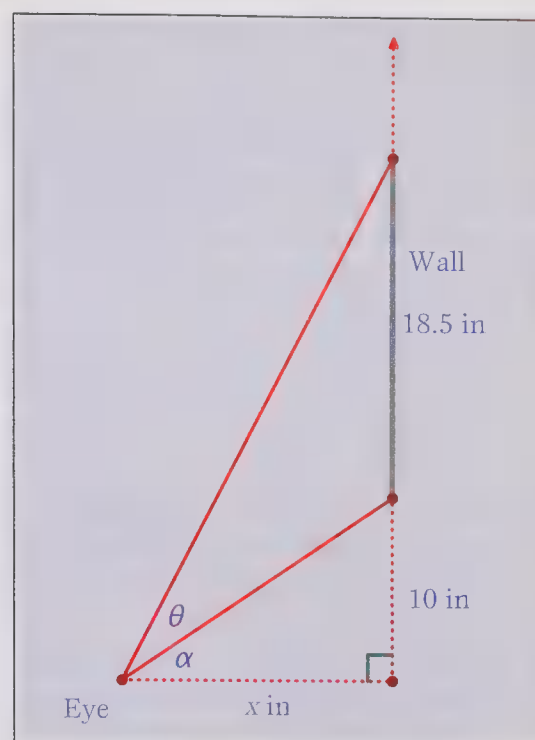


Fig. 2 ("Optimizing HDTV Viewing Angles")

- Use your function to find the distance from the wall that would give the maximum viewing angle. Then compute the maximum viewing angle.
- Which viewing distance would you prefer—the distance that gives the

maximum viewing angle, a distance within the recommended range, or some other distance? Explain.

Onside Kicks

In early 2012, the idea of eliminating kickoffs in the NFL was briefly discussed. A kickoff occurs when one team (the kicking team) kicks the ball to the other team (the returning team) to start the game, after a score, or to start the second half. Most sports pundits declared it a ludicrous idea.

On April 16, 2012, analyst Mark Schlereth appeared on the Mike & Mike show and commented: "What're you

gonna do—flip a coin three times in a row? You gotta get heads three times in a row to get an onside kick?"

An onside kick is a kickoff in which the kicking team tries to recover the ball for itself. An onside kick is successful if the ball travels at least 10 yards and a member of the kicking team recovers the ball before it either goes out of bounds or is caught by a member of the receiving team.



Source: The full interview with Mark Schlereth appears at <http://espn.go.com/espnradio/play?id=7823100>. More details about the event are at Vennebush's blog at <http://mathjokes4mathyfolks.wordpress.com/2012/04/17/kicking-off-flipping-off/>.

- If Schlereth's proposal is adopted, what is the probability that an onside kick would be successful—that is, recovered by the kicking team?
- According to Advanced NFL Stats (www.advancednflstats.com), the probability for success of an onside kick in the NFL is about 26%. Modify Schlereth's proposal so that the outcome would more closely match the actual probability.
- Also according to Advanced NFL Stats, onside kicks are highly dependent on the element of surprise. If the receiving team expects an onside kick, the probability of success is only 20%. Using coin flips, develop a mathematical model for when the receiving team expects an onside kick so that the outcome would approximately match the actual probability.
- If the receiving team does not expect an onside kick, however, the probability of success increases to about 60%. Using coin flips, develop a mathematical model for when the receiving team does not expect an onside kick so that the outcome would approximately match the actual probability.

"Optimizing HDTV Viewing Angles" answers

1. The function is given by

$$\theta(x) = \arctan\left(\frac{18.5}{x}\right),$$

where $\theta(x)$ represents the viewing angle (in degrees) and x represents the viewer's distance from the wall (in in.). The domain of the function is given by $x > 0$ because the arctan function is continuous for all real $x \neq 0$ and the viewer's distance from the wall cannot be negative in this situation.

2. For the viewing distances of 4.63 ft. and 7.71 ft., the viewing angles would be only about 18.4° and 11.3° . Convert 4.63 ft. to 55.56 in. to get

$$\theta(55.56) = \arctan\left(\frac{18.5}{55.56}\right) \approx 18.4^\circ.$$

Convert 7.71 ft. to 92.52 in. to get

$$\theta(92.52) = \arctan\left(\frac{18.5}{92.52}\right) \approx 11.3^\circ.$$

3. There does exist an upper limit to the viewing angle. As the viewer gets closer to the screen—that is, as the viewing distance decreases to 0—the viewing angle increases to 90° :

$$\lim_{x \rightarrow 0} \arctan\left(\frac{18.5}{x}\right) = 90^\circ$$

4. The function is given by

$$\theta(x) = \arctan\left(\frac{10+18.5}{x}\right) - \arctan\left(\frac{10}{x}\right),$$

where $\theta(x)$ represents the viewing angle (in degrees) and x represents the viewer's distance from the wall

(in in.). The domain of θ is given by $x > 0$ because both arctan functions are continuous for all real $x \neq 0$. Therefore, their difference is also continuous for $x \neq 0$, and the viewer's distance from the wall cannot be negative in this situation.

5. For the two given distances, 4.63 ft. and 7.71 ft., the viewing angles would be, respectively, about 17.0° and about 11.0° .

Convert 4.63 ft. to 55.56 in. to get

$$\begin{aligned} \theta(55.56) &= \arctan\left(\frac{28.5}{55.56}\right) \\ &\quad - \arctan\left(\frac{10}{55.56}\right) \approx 17.0^\circ. \end{aligned}$$

Convert 7.71 ft. to 92.52 in. to get

$$\begin{aligned} \theta(92.52) &= \arctan\left(\frac{28.5}{92.52}\right) \\ &\quad - \arctan\left(\frac{10}{92.52}\right) \approx 11.0^\circ. \end{aligned}$$

6. The maximum viewing angle would be about 29° and would occur at a viewing distance of about 17 in. from the wall. Consider the following:

$$\begin{aligned} \theta'(x) &= \frac{\frac{-28.5}{x^2}}{1 + \left(\frac{-28.5}{x^2}\right)^2} - \frac{\frac{-10}{x^2}}{1 + \left(\frac{-10}{x^2}\right)^2} \\ &= \frac{-28.5}{x^2 + 28.5^2} + \frac{10}{x^2 + 10^2} \\ &= \frac{-28.5(x^2 + 10^2) + 10(x^2 + 28.5^2)}{(x^2 + 28.5^2)(x^2 + 10^2)} \\ &= \frac{-18.5(x - \sqrt{285})(x + \sqrt{285})}{(x^2 + 28.5^2)(x^2 + 10^2)} \end{aligned}$$

All factors in the last expression are positive except $-18.5(x - \sqrt{285})$. This product is positive for $x < \sqrt{285}$ and negative for $x > \sqrt{285}$, so θ increases to the left and decreases to the right of $x = \sqrt{285} \approx 16.88$ in. Therefore, the maximum value of θ occurs at $x = \sqrt{285}$ and is given by

$$\begin{aligned} \theta(\sqrt{285}) &= \arctan\left(\frac{28.5}{\sqrt{285}}\right) \\ &\quad - \arctan\left(\frac{10}{\sqrt{285}}\right) \approx 28.7^\circ. \end{aligned}$$

7. Answers may vary. In this scenario, a viewer would have to sit awfully close to the wall (about 17 in.) to achieve the maximum viewing angle (about 29°). Sitting this close to an HDTV could result in neck or back pain, eyestrain, or distorted views. A viewer would probably be more comfortable sitting within the recommended distances (between 4.63 ft. and 7.71 ft.) or farther, despite having a smaller viewing angle (between 11° and 17° or smaller).

"Onside Kicks" answers

1. The probability that an onside kick would be successful is .125:

$$P(3 \text{ heads}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} = .125$$

2. We could flip two coins instead of three. Then

$$P(2 \text{ heads}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = .25,$$

which is close to .26.

3. We find the probability of attaining n heads when 9 coins are flipped:

Table 1 ("Onside Kicks")

n	0	1	2	3	4	5	6	7	8	9
$P(n)$	.001953	.017578	.070313	.164063	.246094	.246094	.164063	.070313	.017578	.001953

${}_9C_n \cdot (1/2)^9$. The probabilities for 0–9 heads are shown in **table 1** (“Onside Kicks”). Consequently, one option is to flip nine coins, and if 1, 3, 8, or 9 heads occur, the kick is successful. The probability of one of those occurring is $P(1, 3, 8, \text{ or } 9 \text{ heads with } 9 \text{ coins}) = .017578 + .164063 + .017578 + .001953 = .201172$, or about 20%.

4. Again, flip 9 coins. As shown in **table 1** (“Onside Kicks”), if 1, 2, 4, 5, 8, or 9 of them land heads, the onside kick is successful. The probability of one of those occurring is $P(1, 2, 4, 5, 8, \text{ or } 9 \text{ heads with } 9 \text{ coins}) = .017578 + .070313 + .246094 + .246094 + .017578 + .001953 = .599609$, or about 60%.



MARGARET CIBES, cibesm@comcast.net, and **JAMES GREENWOOD**, jhgreenwoodjr@gmail.com, who submitted “Optimizing HDTV Viewing Angles,” are retired mathematics teachers. Their main interest lies in the promotion of quantitative literacy through analysis of the mathematical and statistical reasoning found in a variety

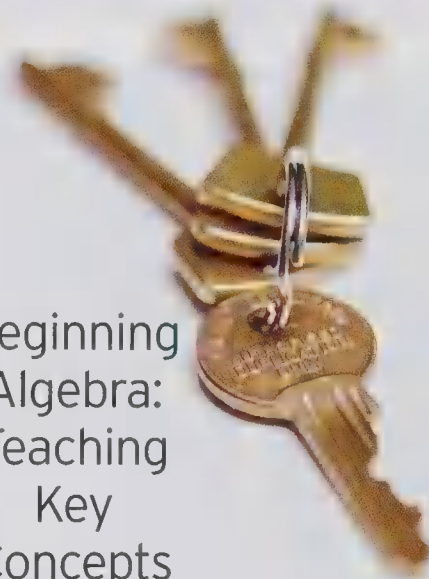


of media. **G. PATRICK VENNEBUSH**, patrick.vennebush@verizon.net, who submitted “Onside Kicks,” is the director of mathematics for Discovery Education, where he is leading the development of a mathematics techbook.

COMING IN
NOVEMBER

MT 2013 FOCUS ISSUE

Beginning
Algebra:
Teaching
Key
Concepts



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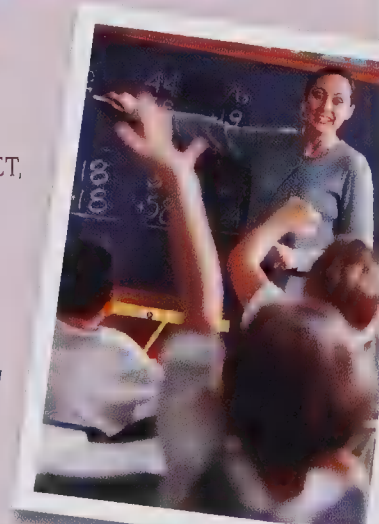
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MATHEMATICS
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Horsing Around



Photograph 1 The carousel at
Sandwich, Massachusetts

MARY HAYNES

Carousels, or merry-go-rounds, have been a mainstay of amusement parks for years, entertaining young and old alike. In addition to providing a link to an earlier era, carousels provide opportunities to engage in some interesting mathematics.

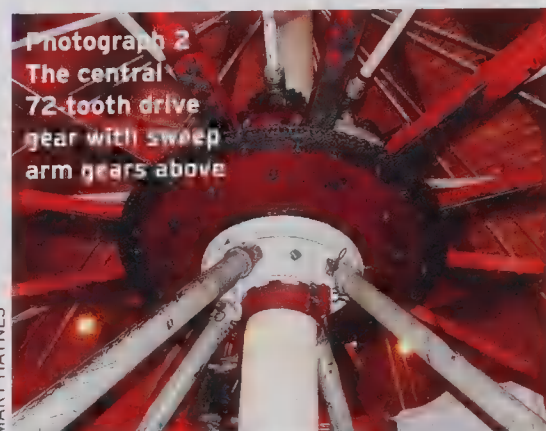
The carousel at Heritage Museums and Gardens in Sandwich, Massachusetts

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Edited by **Ron Lancaster**, ron2718@nas.net, University of Toronto, ON, Canada, and **Brigitte Bentele**, brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY

(see **photograph 1**), was built in 1908 in East Providence, Rhode Island, by the Charles Looff Company. The carousel measures 35 feet in diameter and makes one revolution every 14 seconds. In each revolution, a rider on one of the horses experiences four up-and-down cycles.

1. The two motions of the carousel—the circular motion of the entire carousel and the rising and falling of individual horses—are controlled by a system of three gears. The first gear is the electric motor shaft gear, a 12-tooth gear that transfers power from the motor to the carousel. The central drive gear is a 72-tooth gear that is connected to the electric motor shaft gear by a chain (see **photograph 2**); this gear provides the rotational motion of the carousel. The “sweep” arm gear is an 18-tooth gear that provides the vertical motion



Photograph 2
The central
72-tooth drive
gear with sweep
arm gears above

MARY HAYNES

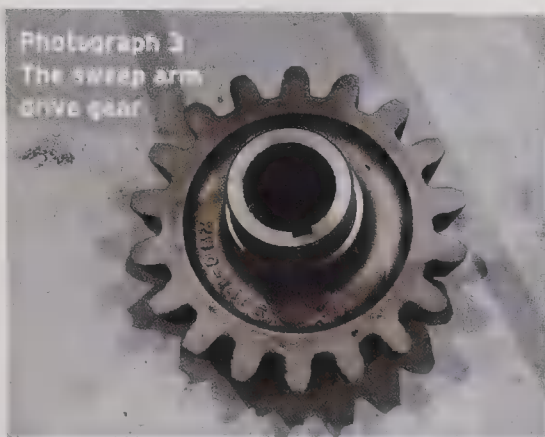
to the horses. The gear is shown in **photograph 3**, and the sweep arm is shown in **photograph 4**. A total of 12 sweep arms radiate from the center of the carousel.

All gears work in a 1:1 ratio. An advance of 1 tooth in the electric motor shaft gear advances the central drive gear 1 tooth; this action, in turn, advances the sweep arm gear 1 tooth.

(a) In each revolution of the carousel, the rider experiences four vertical cycles on the horse. The sweep rotates 4 times to produce 4 ups and four downs per revolution. Thus, each 1-tooth advance of the sweep gear represents how many degrees of a carousel rotation?

(b) The gears for the sweep are driven by the central 72-tooth drive gear, which in turn is driven by the 12-tooth gear on the electric motor shaft. Each time the gear on the electric motor shaft makes 1 complete revolution, how many degrees does the carousel advance?

2. (a) The motor and drive shafts are housed in the center of the carousel. As a result, it is almost impossible to determine the radius of



MARY HAYNES



MARY HAYNES



MARY HAYNES

each of the three rings of horses that make up the carousel. However, there are 12 sweep arms, so the angle between consecutive sweep arms is 30° . We also know the distance between horses for each ring (see **fig. 1**). For the inner ring, the distance between horses is 62 in.; for the middle ring, the distance is 77 in.; and for the outer ring, the distance is 93 in. Use this information to calculate, to the nearest half foot, the radius of each of the three rings.

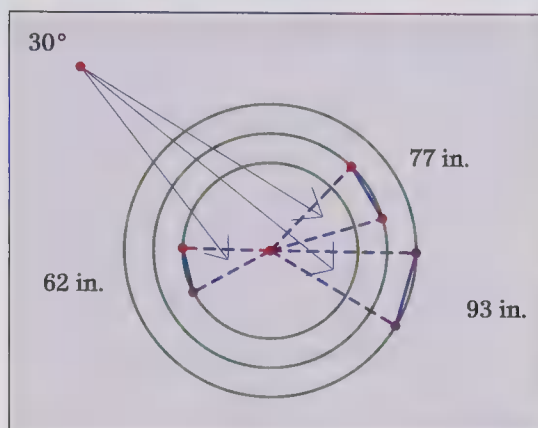


Fig. 1 The distance between horses varies for each ring of the carousel.

of 54 in. Sketch one period of the horse's up-and-down motion, with time measured in seconds on the x -axis and saddle height measured in inches on the y -axis.

(b) Using radians, write an equation for the horse's height in inches as a function of time.

(c) If the horse begins a 2-minute ride at its maximum saddle height, what is the saddle height at the end of the ride?

4. Visitors love to take pictures and videos of their children riding the carousel. Suppose that an observer with a stationary, motor-driven video camera is positioned 5 ft. from the closest position of a particular rider on the outer ring of the carousel. To keep this rider in view for the video camera, the observer would need to consider the changing distance between the rider and camera as well as the changing angle.

(a) **Figure 2** shows the video camera at point A and the rider at point B. Determine an equation for the

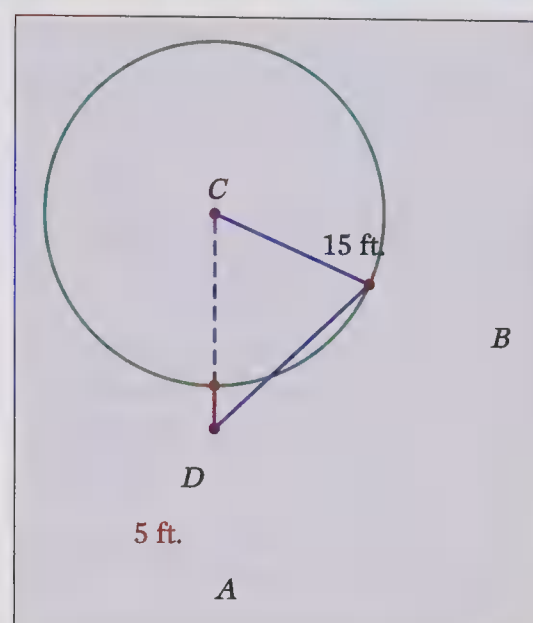


Fig. 2 An observer is at point A. The horse and rider in the outer ring start at point D and travel in the direction of B.

distance, as a function of time, between the camera and rider. Assume that at time $t = 0$ the horse and rider are at the closest point to the observer, at point D, and that the direction of rotation is counterclockwise, from D to B.

(b) Is this distance function periodic? Is its graph sinusoidal? Explain.

(c) Determine an equation for the angle ($\angle CAB$), as a function of time, that the camera would need to make to keep the rider in view. Let angles to the right of segment AC be considered positive and those to the left negative. Obtain a graph of this function.

(d) Explain why the graph of the angle function has the shape that it does. Refer to the rider's circular motion on the carousel and the camera's stationary position.

(b) Without including the up-and-down movement of the horse and rider, determine the distance traveled along each circular ring during a typical ride lasting 2 minutes. Remember that the carousel makes 1 revolution every 14 seconds.

(c) The horse and rider are not at the same location at the beginning and at the end of a typical ride, whose duration is 2 minutes. If the radius from the center of the carousel to the horse and rider at the start of the ride is the initial side of an angle and if the radius from the center to the horse and rider at the end of the ride is the terminal side, what angle is made by the two radii? Again, ignore the vertical motion of the horse and rider.

3. (a) The height of the horse is controlled by the upside-down U-shape portion of the sweep arm (see **photograph 4**). Each horse's pole is moved up and down by the 6-in. offset from the shaft. The horse shown in **photograph 5** is at its maximum saddle height

MATHEMATICAL LENS solutions

- (a) The sweep arm gear has 18 teeth, so 1 complete revolution of the carousel would involve 4 vertical cycles, or 4 revolutions of the sweep gear. Thus, there would be 72 advances of the teeth, giving an angle advance of 5° for the carousel.
- (b) The 12-tooth gear aligned with the motor would drive the 72-tooth main gear $1/6$ of 1 revolution of the carousel. Thus, 1 revolution of the electric motor gear would correspond to an angle of 60° .

- (a) The inner ring of the carousel is shown in **figure 3**. We can use either right-triangle trigonometry or the law of cosines to determine each radius.

In triangle H_1H_2C , the measure of angle $H_1CM = 15^\circ$. If we use the given measures for the distances between horses, $H_1M = 62/2 = 31$ in. Then

$$\sin 15^\circ = \frac{31}{r}$$

$$r \approx 119.77 \text{ in.}$$

If we convert to feet, the inner radius is approximately 10 ft.

Let's use

$$\sin 15^\circ = \frac{\frac{77}{2}}{r}$$

for the middle ring and

$$\sin 15^\circ = \frac{\frac{93}{2}}{r}$$

for the outer ring. Converting to feet and rounding to the nearest half foot, we find that the middle ring has a radius of 12.5 ft. and the outer ring has a radius of 15 ft.

We can also apply the law of cosines to triangle H_1H_2C , as follows:

$$62^2 = 2r^2 - 2r^2 \cos(30)$$

$$62^2 = r^2(2 - \sqrt{3})$$

$$r^2 = \frac{62^2}{2 - \sqrt{3}}$$

$$r \approx 119.77 \text{ in.}$$

- (b) Because each revolution takes 14 sec., a 2-min., or 120-sec., ride gives $120/14 = 60/7$, or $8 \frac{4}{7}$, revolutions. If we use 10 ft. as the radius of the inner ring, the circumference is 20π ft., so the total

distance traveled is about 539 ft. For the middle ring, the total distance is approximately 673 ft., and for the outer ring it is about 807 ft.

- (c) Because the horse ends up $4/7$ of a revolution past its starting point, the angle formed with the starting radius as the initial side of the angle and the ending radius as the terminal side is about 206° .

- (a) There are 4 vertical cycles in 1 revolution of 14 sec., so the horse reaches its maximum height of 54 in. every 3.5 sec. Since the arm sweeps up and down 6 in. from the shaft, the minimum height of the horse is $54 - 12 = 42$ in. The graph in **figure 4** shows five key points in one period of the horse's motion.

- (b) If we measure in radians, the height in inches as a function of time may be represented by

$$H(t) = 6 \cos\left(\frac{4\pi}{7}t\right) + 48.$$

The sinusoidal axis is halfway between the maximum of 54 and the minimum of 42, so we know

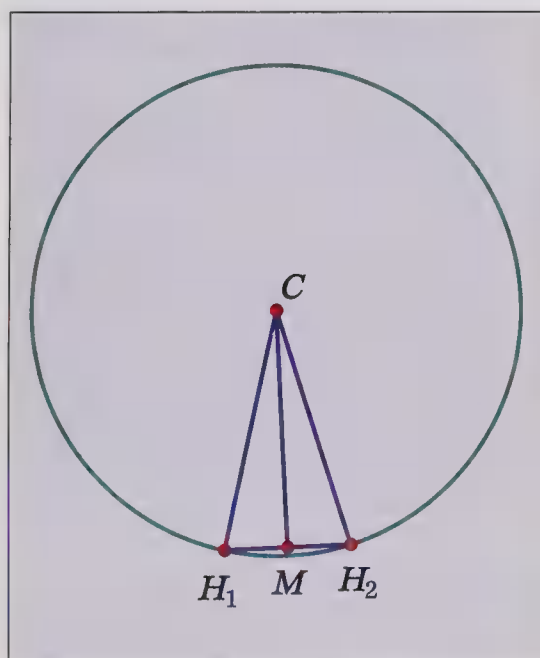


Fig. 3 The distance between the horses determines the radius.

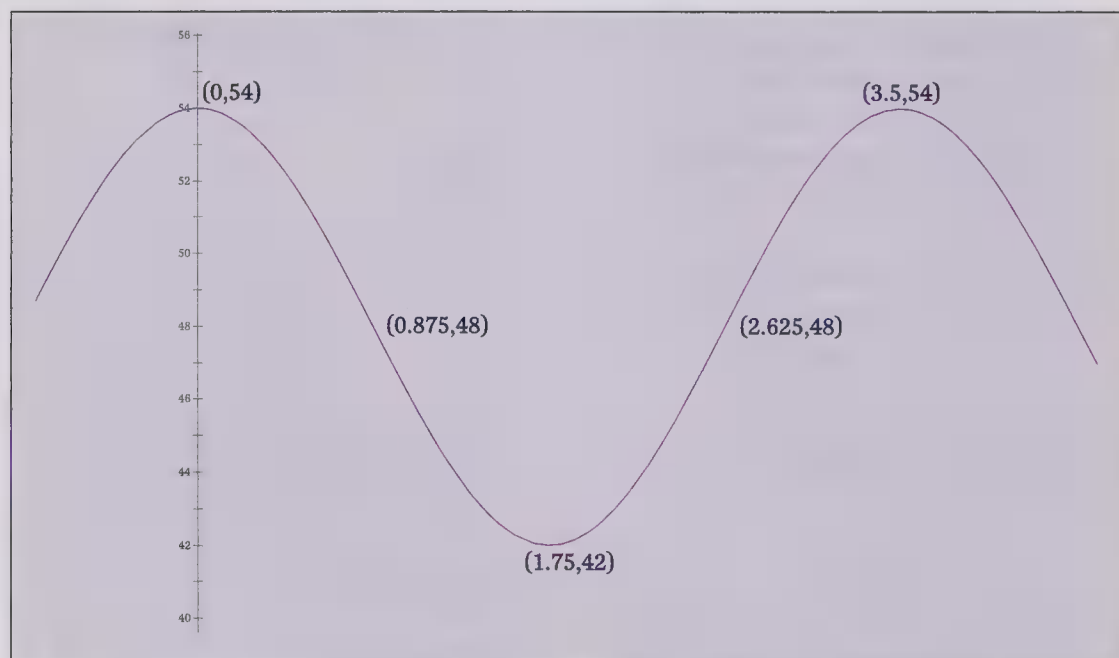


Fig. 4 A horse's saddle height in inches can be graphed as a function of time measured in seconds.

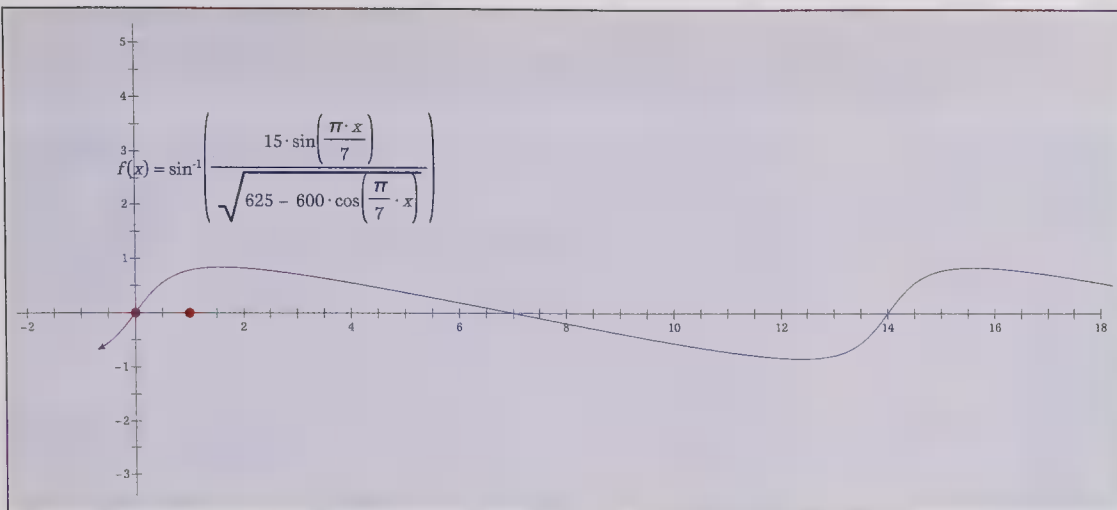


Fig. 5 The graph shows the camera angle in radians as a function of time.

that the curve is shifted up 48. The amplitude is half the difference of the maximum and minimum heights, or 6. We also know that the period is $t = 3.5$, meaning that the frequency is $2\pi/3.5 = 4\pi/7$. Using the cosine curve enables us to avoid a horizontal shift, but a variety of sine functions would work as well. One such function is

$$H(t) = 6 \sin\left(\frac{4\pi}{7}(t - 2.625)\right) + 48.$$

(c) Use one of the functions obtained in part 3(b) evaluated at $t = 120$ sec. to find that the saddle height at the end of the ride would be around 46.66 in., or 3.89 ft.

4. (a) Using the law of cosines, we have the following:

$$AB^2 = 20^2 + 15^2 - 2 \cdot 20 \cdot 15 \cos\left(\frac{\pi}{7}t\right)$$

$$AB^2 = 625 - 600 \cdot \cos\left(\frac{\pi}{7}t\right)$$

$$AB = \sqrt{625 - 600 \cdot \cos\left(\frac{\pi}{7}t\right)}$$

(b) The function is periodic with a period of 14 sec., but because the function is a composition of the square root function and a trigonometry function, it is not a sinusoid. The square of the distance, however, is a sinusoid.

(c) We can use the law of sines to determine $\angle CAB$:

$$\frac{\sin(\angle CAB)}{CB} = \frac{\sin(\angle ACB)}{AB}$$

$$\sin(\angle CAB) = \frac{15 \cdot \sin\left(\frac{\pi}{7}t\right)}{\sqrt{625 - 600 \cos\left(\frac{\pi}{7}t\right)}}$$

$$m\angle CAB = \sin^{-1}\left(\frac{15 \cdot \sin\left(\frac{\pi}{7}t\right)}{\sqrt{625 - 600 \cos\left(\frac{\pi}{7}t\right)}}\right)$$

The graph of the camera angle as a function of time is shown in **figure 5**.

(d) The maximum and minimum camera angles are about 0.848 radians, or 48.6° , and -0.848 radians, or -48.6° . The camera will have to reach its maximum

angle very quickly when a ray from the camera is tangent to the ring the horse travels on and would be perpendicular to the centerline to that point. The camera angle at that point is 0.848 radians, or 48.6° . Thus, $m\angle ACB = 0.723$ radians, or 41.4° . Since that central angle has a measure of $\pi t/7$ radians, the time to reach that point is 1.61 sec. The camera would have to move very quickly as the horse starts its revolution, then pan slowly left until it reaches its minimum angle (maximum negative angle), and then quickly come back to 0 over the last 1.61 sec. of the 14-sec. revolution around the carousel—quite a challenge for someone photographing a horse and rider on a carousel.

ACKNOWLEDGMENTS

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DOUGLAS WILCOCK,

dougw@capecodacademy.org, recently retired from Cape Cod Academy in Osterville, Massachusetts. He continues to be interested in finding mathematics in the world around him.

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More Than Numbers:

The early grades set the foundation for students' success in mathematics in later grades. In these early grades, children develop the capacity to handle complex and sophisticated mathematics; become confident that they can attack problems; and learn to build a tool set of generalizations, skills, and concepts that can be used as they progress. We, as mathematics teachers, parents, and administrators, must regularly ask ourselves:

- How can we help students acquire these generalizations, skills, and concepts?
- How can we create mathematical experiences that engage young children while building robust understanding?
- How can we help young students understand number concepts and learn to count and compute proficiently?
- How can students not only learn facts about shapes, patterns, and measurement but also find generalizations that will help them solve more complex problems?

Barbara Dougherty, Ph.D., University of Missouri–Columbia; Amy Lingo, Ed.D., University of Louisville; and Lynn Columba, Ph.D., Lehigh University, answer all these questions in NCTM's new response-to-intervention (RtI) digital series Math of Tomorrow. Also known as MOTO, Math of Tomorrow is a series of 7 interactive books available on iTunes, GooglePlay, Amazon, and the NCTM website, www.nctm.org/moto.

- *Clock Wise: Learning Time with the MOTO Family*
- *What Comes Next? Making Patterns with the MOTO Family*
- *Shapes and Sizes: Learning Geometry with the MOTO Family*
- *One Foot, Two Feet: Measuring with the MOTO Family*
- *Everybody Counts: Learning to Count with the MOTO Family*
- *It All Adds Up! Learning to Add and Subtract with the MOTO Family*
- *Bits and Pieces: Sorting and Representing Data with the MOTO Family*

Engage Young Children

Creating high interest in mathematical concepts is key to helping students who struggle. Their engagement in tasks builds their understanding as well as their interest. In the MOTO series, tasks are—

- **Embedded in stories.** Children work through mathematical situations as they see how each member of the MOTO family thinks about mathematics. For example, Addison uses reasoning while Multiplex uses guess and test. Mathematical situations are embedded in MOTO family stories as the characters build a playhouse, make gift boxes of gollywomples for their friends, take inventory in a pet store, or explore different types of clocks. The problems are linked to story contexts so that children attach meaning to concepts, algorithms, and vocabulary.
- **Linked to interactive explorations.** Many students become passive learners in the classroom. Traditional paper-and-pencil tasks do not allow them to self-check or to practice concepts and



Addison

skills as many times as they want. MOTO allows children to complete tasks as many times as they need or like.

- **Given in video and text contexts or stories.** Reading is an important skill for young children. However, providing only text-based materials may not be motivating. The video storyline creates a companion to the text so that students have more motivation for reading.
- **Included in game formats.** Students have opportunities to learn and develop skills through games. For example, students compete against the MOTO family in the Card Shark game on page 124 of *It All Adds Up!* The object is to use 4 numbers to create 2 two-digit numbers such that the difference is close to a target number. As students play, they develop estimation and computational skills. In other games, such as the Matching Game, students play individually.



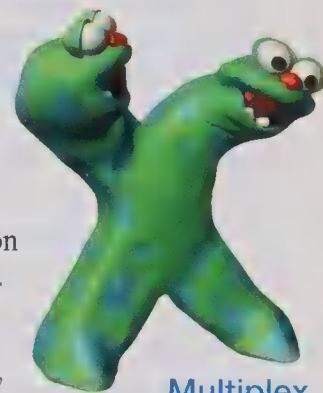
Develop Number Concepts and Skills

Building blocks for later mathematical experiences start with an understanding of counting and number flexibility. Number flexibility, the ability to see a quantity in different ways, builds proficiency with computational algorithms.

Number concept and skill tasks include opportunities to predict a reasonable count, count forward from 1 to X or backward from X to 1, and think of numbers related to groups of 10.

Students should be able to predict by inspection (without counting) if a collection of objects is more than 10 or more than 20. Counting forward and backward leads to counting-on and counting-off strategies for addition and subtraction. In *Everybody Counts*, students have multiple opportunities to practice these techniques.

Thinking of quantities as groups of 10 helps children develop robust algorithms for computations and an understanding of place value. For example, in *It All Adds Up!*, Mina uses a strategy of doubles subtract 1 to find a sum. In the interactive activities, students explore multiple strategies for computations; some are mental strategies, and others can be used in paper-and-pencil tasks.



Multiplex

Connect across Mathematical Topics

Rather than seeing mathematics as a collection of isolated facts, algorithms, and vocabulary, children who struggle in mathematics should form links across multiple concepts and skills.

In the MOTO series, children see connections across topics that at first glance may appear to be unrelated. For example, in *One Foot, Two Feet*, the concept of unit is developed in measurement contexts. Students come to understand that the size of the unit affects the measure or count. If 2 units are each used to measure length, the count or measure will be different depending on the size of the unit. The smaller unit will produce a larger count or measure because it takes more of them to measure. This is related to counting a set or collection of objects. The count changes depending on the unit used.

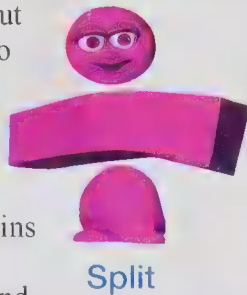
MOTO Helps Children Who Struggle

Make Generalizations

By identifying patterns and generalizations, students can apply their understandings to other concepts and skills. Generalizations help students become proficient and retain a skill longer. If they forget a skill, generalizations allow them to recapture it.

Tasks can promote generalizations. In *Bits and Pieces*, students explore ways to classify shapes. Students develop rules about shapes that are included or excluded from a group and are asked to explain how they created the groups. From the rules they used, they form generalizations about shapes. For example, if they put shapes that have 4 sides in a group, these shapes also have 4 angles.

In *It All Adds Up!*, students create tables to help identify patterns. For example, as the number of green gollywomps decreases by 1, the number of blue gollywomps increases by 1 but the total remains the same. This is a powerful concept for addition—when one addend decreases by 1 and the other addend increases by 1, the sum remains the same.



Split

Support Teachers

Supporting students who struggle requires explicit instructional techniques, such as focusing students' attention on particular aspects of a problem. Rather than telling students each step in the solution, scaffolding questions can help students become self-sufficient and confident problem solvers. Each page of every MOTO book explores—

- **Common Misconceptions.** As students encounter new concepts and skills, it is important to predict possible misconceptions or challenges. These have been identified so that teachers can be prepared.
- **Strategies and solutions.** Many tasks have multiple solution paths. For example, in *What Comes Next?*, the many ways to describe a pattern are described or illustrated for teachers.
- **Scaffolds.** Modeling allows students to see a method or process by which to gain a new skill. In *One Foot, Two Feet*, for example, students have additional support to learn how to directly compare lengths.
- **Questions.** Suggestions for teacher prompts extend beyond factual questions (such as, What is the sum of 8 and 4?) to questions that push students to think more deeply. For example, asking students to “find 3 pairs of numbers whose sum is 12” allows multiple solutions while giving *all* students the opportunity to respond.
- **Extensions.** Some students need additional time or more opportunities to learn. To support these students, additional resources are available in an ancillary product called Teacher Connect, a website that allows teachers to see student work and correlate the lessons to the Common Core State Standards for Mathematics, available at www.nctm.org/moto. These resources include conceptual and skill-based tasks. Additional tasks in the teacher notes can be used to extend students' learning opportunities.



Mina

MOTO's Card Shark Game helps students who struggle to develop estimation and computational skills.

Thinking of quantities as groups of 10 helps children develop algorithms for computations and an understanding of place value.

Each book is aligned with the Common Core State Standards for Mathematics.

Are All Infinity Created

Teo J. Paoletti

Can one infinity be more than another infinity?

Ask students this question, and many will be puzzled; others will insist that “infinity is infinity.” The question seems to pique their interest and provides an opportunity to present the beautifully simple but counterintuitive proofs concerning the size of infinity first constructed by Georg Cantor.

Such a question may not appear to fit within certain specific content, but the concept of infinity emerges within many contexts throughout the secondary school curriculum, from first-year algebra to BC Calculus. When graphing, students must think about what happens as a function goes toward positive and negative infinity or what it means for an equation to have an infinite number of solutions. In calculus, students begin to think about getting infinitely close to a point without ever getting there. The idea of infinity in relation to these topics is often glossed over in a procedural treatment that ignores the possibility of allowing students to experience deeper levels of mathematical reasoning.

Two other topics in the secondary school curriculum closely related to infinity are one-to-one

and onto functions, which have clear connections to Cantor’s work. A function is *one-to-one* if each output value is the image of a unique input; a function is *onto* if each element in the range is an output for some input value from the domain. Too often, students get little exposure to one-to-one and onto functions. Examples typically include a finite list of numbers in a domain and range with arrows connecting these values for students to interpret. **Figure 1** is an example from an online teacher resource manual.

The approach shown in **figure 1** ignores the underlying importance of infinity in deciding whether a continuous function is indeed one-to-one or onto. Instead, students may be introduced to the horizontal and vertical line tests, which they can use on graphs to check whether continuous functions are one-to-one, onto, or both. This type of presentation ignores not only the underlying importance of infinity but also the historical development of the concepts of one-to-one and onto. These concepts relied on important and once-challenged assumptions, transforming many ideas about the concept of infinity and leading to surprising results.

ties Equal?

*This historical—
and classroom
friendly—
approach to
the concept of
infinity uses
Cantor's proofs
of cardinality.*



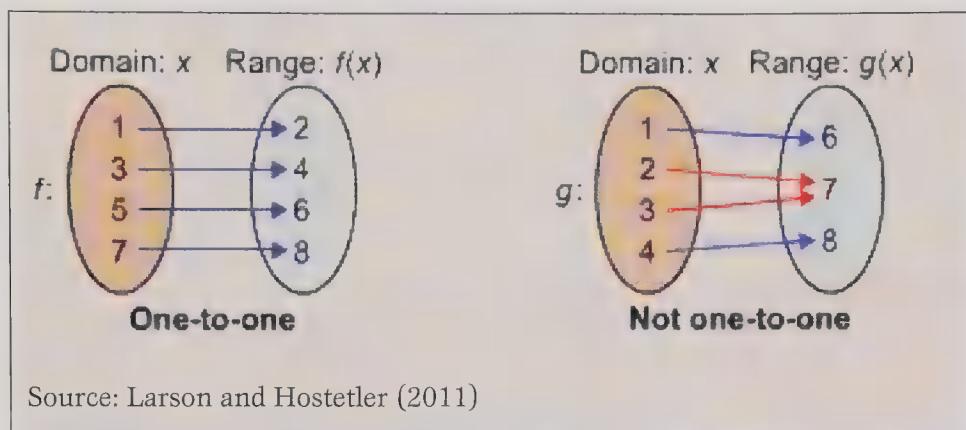


Fig. 1 A common textbook approach to one-to-one functions represents a relationship between two finite sets, an approach that tends to ignore the underlying significance of infinity.

The significance of infinity is supported by NCTM's Standards (2000), which underscore the importance of developing a deep understanding of very large numbers as well as comparing and contrasting the properties of different number systems at the secondary school level. Further, these Standards emphasize the importance of students analyzing and evaluating the mathematical thinking, arguments, and proofs of others. A historical exploration of Cantor's proofs concerning infinite sets is one way in which teachers can develop students' understanding of different sets of numbers while allowing students to evaluate and critique the mathematical arguments of others.

A BRIEF HISTORY

Georg Cantor (1845–1918) worked in many branches of mathematics, although his most important contributions came in the field of set theory. Cantor is the father of set theory, which has had an effect on almost every branch of mathematics (Eves 1990). Cantor both asked and answered mathematics questions never before asked, and his work is sometimes said to be the first original mathematics since the ancient Greeks (Dunham 1991).

One of Cantor's most important insights was to conceptualize an infinite set, such as the real numbers or integers, as a mathematical object whose properties can be examined even though constructing the entire set is impossible (Berlinghoff and Gouvêa 2002). Two prominent mathematicians of the time, Carl Gauss and Leopold Kronecker, both objected to this notion. Gauss believed that one could speak of infinity only as an imaginary idea—not as a mathematical object—and disputed that an infinite set of objects could be considered a complete one (Dunham 1991). Kronecker believed in mathematical objects that could be constructed only in a finite number of steps; thus, he did not consider sets of infinite numbers to be mathematical objects (Berlinghoff and Gouvêa 2002). Kronecker viewed Cantor's concept of a complete

infinite set dangerous and attacked this idea throughout Cantor's life (Berlinghoff and Gouvêa 2002).

Despite the criticisms, Cantor held fervently to his ideas about infinity and set out to devise a way to compare these infinite sets (Dunham 1991). With finite sets, we can simply count the number of elements in each set to determine whether they have the same number of elements. However, with infinite sets, Cantor declared that two sets were equivalent if there was a rule that related every element of one set to a unique element of the other set and vice versa (Dunham 1991). In Cantor's work, the idea of a law or relation did not require a function definition but was simply an organized and systematic way to ensure that all the elements of one set could be matched with all the elements in another set. One way to visualize this concept is to establish a relationship so that each infinite set can march side by side to infinity, regardless of the individual components of each set; then the two sets are equal in size.

In modern terminology, we would say that this relationship is both one-to-one and onto, although Cantor phrased this relationship as a one-to-one correspondence between the two sets. When two infinite sets have a one-to-one correspondence, we say that they have the same cardinality. Armed with the idea of one-to-one correspondence and the premise that infinite sets are complete objects, Cantor explored the cardinality of different sets, resulting in amazing and unintuitive results. What follows are examples and proofs similar to those offered by Cantor.

THE MATHEMATICS OF INFINITY

First, let's explore the relationship between the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ and the set of positive even integers $E = \{2, 4, 6, \dots\}$. Intuition tells us that because the set of natural numbers contains all the positive even integers, there should be more natural numbers than there are positive even integers. Further, students may argue that the natural numbers have twice as many numbers when compared with the positive even integers because the natural numbers are made up of the positive even and positive odd integers equally. However, we can match each integer with its double: 1 corresponds to 2, 2 corresponds to 4, 3 corresponds to 6, and so on. In doing so, we create a mapping from all the natural numbers to all the positive even integers such that each natural number corresponds to a positive even integer and each positive even integer corresponds to a natural number (see **fig. 2**).

We have established a relationship such that the elements of both infinite sets march side by side to infinity and are therefore equal in size. Connecting

this relationship to algebraic functions, we see that the function $f(n) = 2n$, where n is a natural number, gives a one-to-one correspondence between the natural numbers and the positive even integers, meaning that the two sets have the same cardinality. Hence, there are as many natural numbers as there are positive even integers.

Next, we can compare $\mathbb{N}$ to the integers $\mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$. Because $\mathbb{Z}$ contains the set $\mathbb{N}$ as well as all the negated values of $\mathbb{N}$, it would seem that $\mathbb{Z}$ is at least twice as big as $\mathbb{N}$. However, if we can find an organized way to arrange the two sets so that there is a one-to-one correspondence between them, they must have the same cardinality. Rewriting the integers as $\{0, 1, -1, 2, -2, 3, -3, 4, \dots\}$, we see the following one-to-one correspondence (see **fig. 3**).

Examining **figure 3**, we see that we have created a chart in which the natural numbers and the integers march together toward infinity, meaning that the two sets are the same size. Defining an algebraic function for this relationship may not be easy, but the chart serves as proof that there is a one-to-one correspondence. (The corresponding formula for this relationship is

$$f(n) = \frac{1 + (-1)^n (2n - 1)}{4},$$

where n is a natural number.)

Realizing that our intuition may not be serving us well, we move to comparing $\mathbb{N}$ with $\mathbb{Q}$, the set of all fractions in simplest form. Not only does $\mathbb{Q}$ contain all the natural numbers, but it also contains all fractions. Hence, each natural number will be used as the numerator and the denominator of different fractions, leading many students to guess that these two sets cannot be the same size. Looking for a one-to-one correspondence, we first make an organized list of the rational numbers. To do so, we create a two-dimensional chart (see **fig. 4**) in which the elements of the n th row have n as a denominator; in each column, the numerators correspond to the integer in the numerator of the top row. The exception is 0, which will be paired with the natural number 1.

We see that this chart, continuing both vertically and horizontally, would include all the rational numbers (except 0, which is matched to 1). Now we need to devise a way to put the rational numbers in this chart in a one-to-one correspondence with the natural numbers. To do so, we will follow the pattern shown in **figure 5**.

Start from the cell in the top-left corner, $1/1$, and move down one cell to $1/2$. Next, move along the diagonal up and to the right to $-1/1$. Now move to the right one cell to $2/1$. Moving down and to the left along the diagonal, we reach $1/3$. Now we move

$\mathbb{N}$	1	2	3	4	...
$\mathbb{E}$	2	4	6	8	...

Fig. 2 Pairing natural numbers with their doubles shows that $\mathbb{N}$ and $\mathbb{E}$ have the same cardinality.

$\mathbb{N}$	1	2	3	4	5	6	...
$\mathbb{Z}$	0	1	-1	2	-2	3	...

Fig. 3 A one-to-one correspondence can be established between the natural numbers and the integers.

$\frac{n}{z}$	$\frac{1}{z}$	$\frac{-1}{z}$	$\frac{2}{z}$	$\frac{-2}{z}$	$\frac{3}{z}$	$\frac{-3}{z}$	$\vdots$
$\frac{n}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\vdots$
$\frac{n}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{2}$	$\frac{-2}{2}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\vdots$
$\frac{n}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{3}$	$\frac{-3}{3}$	$\vdots$
$\frac{n}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{4}$	$\frac{-2}{4}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\vdots$
$\frac{n}{5}$	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\vdots$
$\frac{n}{6}$	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{6}$	$\frac{-2}{6}$	$\frac{3}{6}$	$\frac{-3}{6}$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fig. 4 This arrangement can be used to establish a correspondence between $\mathbb{N}$ and $\mathbb{Q}$.

$\frac{n}{z}$	$\frac{1}{z}$	$\frac{-1}{z}$	$\frac{2}{z}$	$\frac{-2}{z}$	$\frac{3}{z}$	$\frac{-3}{z}$	$\vdots$
$\frac{n}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\vdots$
$\frac{n}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{2}$	$\frac{-2}{2}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\vdots$
$\frac{n}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{3}$	$\frac{-3}{3}$	$\vdots$
$\frac{n}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{4}$	$\frac{-2}{4}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\vdots$
$\frac{n}{5}$	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\vdots$
$\frac{n}{6}$	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{6}$	$\frac{-2}{6}$	$\frac{3}{6}$	$\frac{-3}{6}$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Fig. 5 Following this path establishes the correspondence.

down one cell and again move up and to the right along the diagonal again, skipping any repeated values—for example, $2/2 = 1/1$. On eventually reaching the top, move to the right one cell and then move along the diagonal down and to the left.

We are able to create a rule so that the two sets march together toward infinity, meaning that there is a one-to-one correspondence between the natural numbers and the rational numbers. Hence, they have the same cardinality.

At this point, we have established that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all the same size, seemingly contradicting many of our intuitions. So far, only one size, or cardinality, of infinite sets appears to be possible because every set that we have looked at so far has a one-to-one correspondence with the natural numbers.

The next set we look at is the real numbers, which comprise the rational and the irrational numbers. Rather than looking at the entire set of real numbers, we will instead look at the set of all real numbers between 0 and 1 (seemingly a much smaller subset of the real numbers). There is no obvious way to create a one-to-one relationship between this set and the set of natural numbers that gives a one-to-one correspondence. However, if we assume that we have such a correspondence, we notice something interesting.

First, let's assume that we have a one-to-one correspondence between $\mathbb{N}$ and the real numbers between 0 and 1 (see **fig. 6**). The right column of the correspondence shown in **figure 6** is chosen arbitrarily from the infinitely many possible real numbers between 0 and 1. In the n th row, a_1 represents the first digit of the n th decimal expansion, a_2 represents the second digit of the n th decimal expansion, a_n represents the n th digit of the n th decimal expansion, and so on. Because we are assuming a one-to-one correspondence, all real numbers between 0 and 1 will be on this list. However, we can construct a real number not on this list.

To start, we look at the first digit of the real number in the first row of the chart in **figure 6**—in this case, 4. For our new number, we choose any value between 1 and 8 except for 4 (we ignore 9s to prevent a case in which $0.099999999\dots = 0.1$). We

1	0.4926803 . . .
2	0.6919531 . . .
3	0.2500000 . . .
4	0.1838503 . . .
5	0.3348592 . . .
6	0.0128395 . . .
$\vdots$	$\vdots$
n	$0.a_1a_2a_3\dots a_n\dots$
$\vdots$	$\vdots$

Fig. 6 An indirect argument can be used to establish the relationship between the natural numbers and the reals.

will pick 2, and our new irrational number starts as 0.2. For the second digit of our new number, we look to the second digit from the irrational in the second row of the chart—in this case, 9. Again, we pick a number other than 9, so we pick 7, meaning that our new irrational continues to 0.27. For the third digit, we go to the irrational in the third row—in this case, 0—and instead choose 5. Now our new irrational becomes 0.275.

We can continue this procedure for each of the infinite numbers in our list, always making sure that the n th digit (a_n) in our new number is different from the n th digit of the real number in the n th row. Hence, the number we are creating is different from every number on the list, at least in the n th digit. Thus, our new number is not on the list.

Using a proof by contradiction, we have found that there is not a one-to-one correspondence between the natural numbers and the set of real numbers between 0 and 1; thus, the two sets are not the same size. Dunham calls this Cantor's diagonalization process (Dunham 1991).

IN THE CLASSROOM

There are many points in the secondary school curriculum at which teachers can introduce Cantor's ideas and beautiful proofs concerning infinity. One such point is when introducing the concept of one-to-one and onto functions during a series of lessons on inverse functions. Instead of presenting students with a few basic examples of one-to-one functions before teaching the horizontal line test, teachers can ask students which set they believe is larger—the natural numbers or the set of positive even integers. This question can lead to an engaging conversation in which students make various conjectures, some believing that all infinities are the same and others guessing that the natural numbers will be larger. Teachers can then present Cantor's proofs, allowing students to critique and question the arguments along the way.

This approach not only allows students to review concepts concerning sets of different size, explore the idea of one-to-one correspondence, and analyze and critique the arguments of others. It also shows the need for the concepts of one-to-one and onto—namely, the requirement to find relationships between two infinitely large sets.

Exploration of these ideas brings forth and reviews students' understanding of the different sets of numbers—including natural numbers, integers, rational numbers, and irrational numbers—all of which appear in the Common Core State Standards (CCSSI 2010) and NCTM's Standards (NCTM 2000). Further, analysis of these proofs allows students to look for and make use of structure, reason abstractly, and express regularity in repeated

reasoning, all important mathematical practices defined in the CCSSM (CCSSI 2010). Although a unit on inverse functions is one opportunity to introduce these elegant arguments and proofs, it is not the only point at which these proofs would be helpful in piquing students' interest.

Cantor's arguments and proofs were groundbreaking for his time and had far-reaching implications for many types of mathematics. However, they are so simple and elegant that most secondary school mathematics students can understand them. By bringing historically significant proofs into their classrooms, teachers can connect multiple mathematical ideas in powerful ways and lead to other links between mathematics concepts and the history of mathematics.

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TEO J. PAOLETTI, paolett2@uga.edu, who previously taught high school mathematics, is pursuing his PhD in mathematics education at the University of Georgia in Athens. He is interested in student learning, curriculum development, and secondary and postsecondary school mathematics.

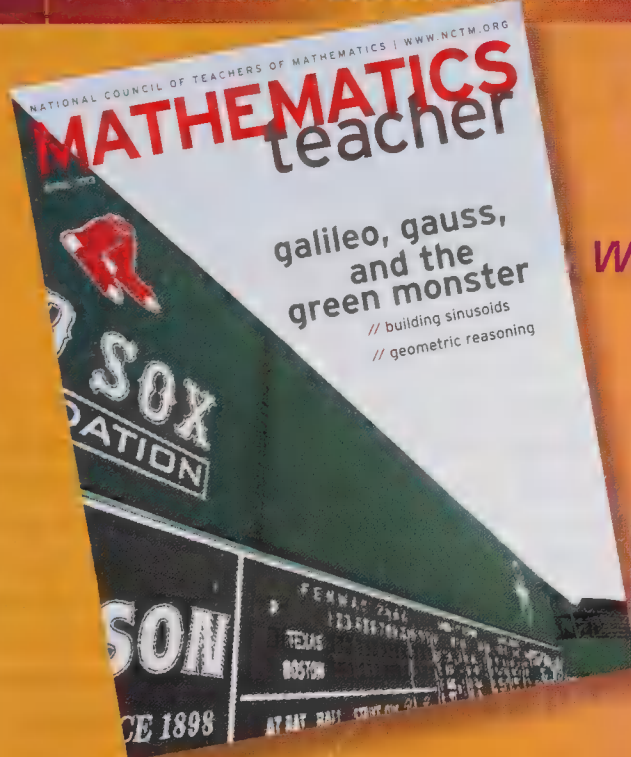
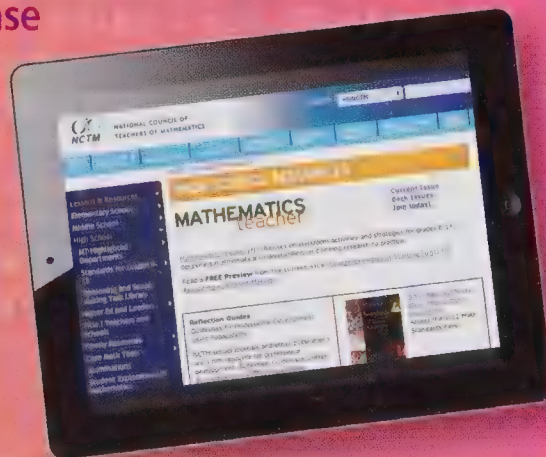
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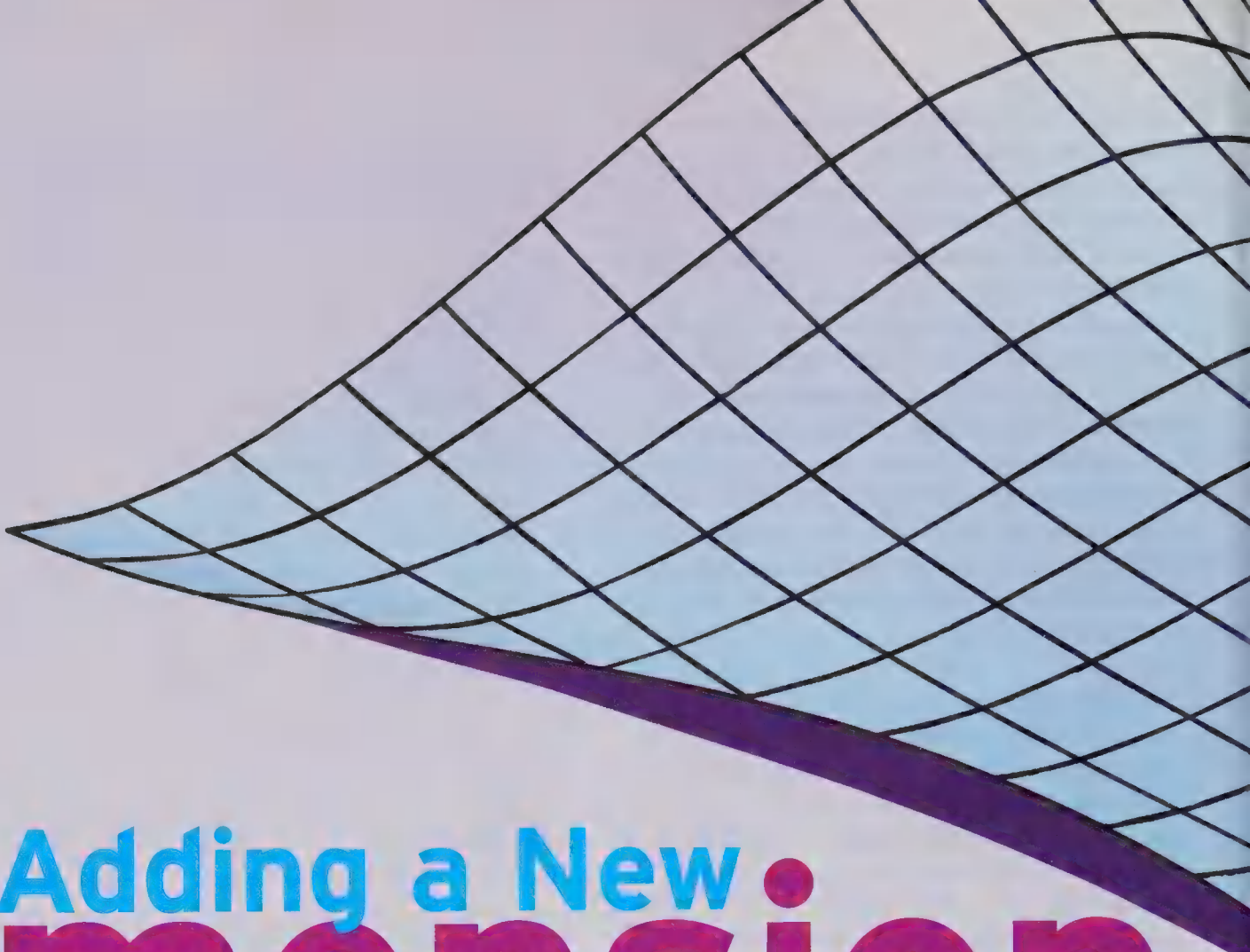
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• Adding a New • Dimension to Algebra

Sheldon P. Gordon

The idea of a function of several variables—important in other disciplines—can be introduced in secondary school mathematics courses.

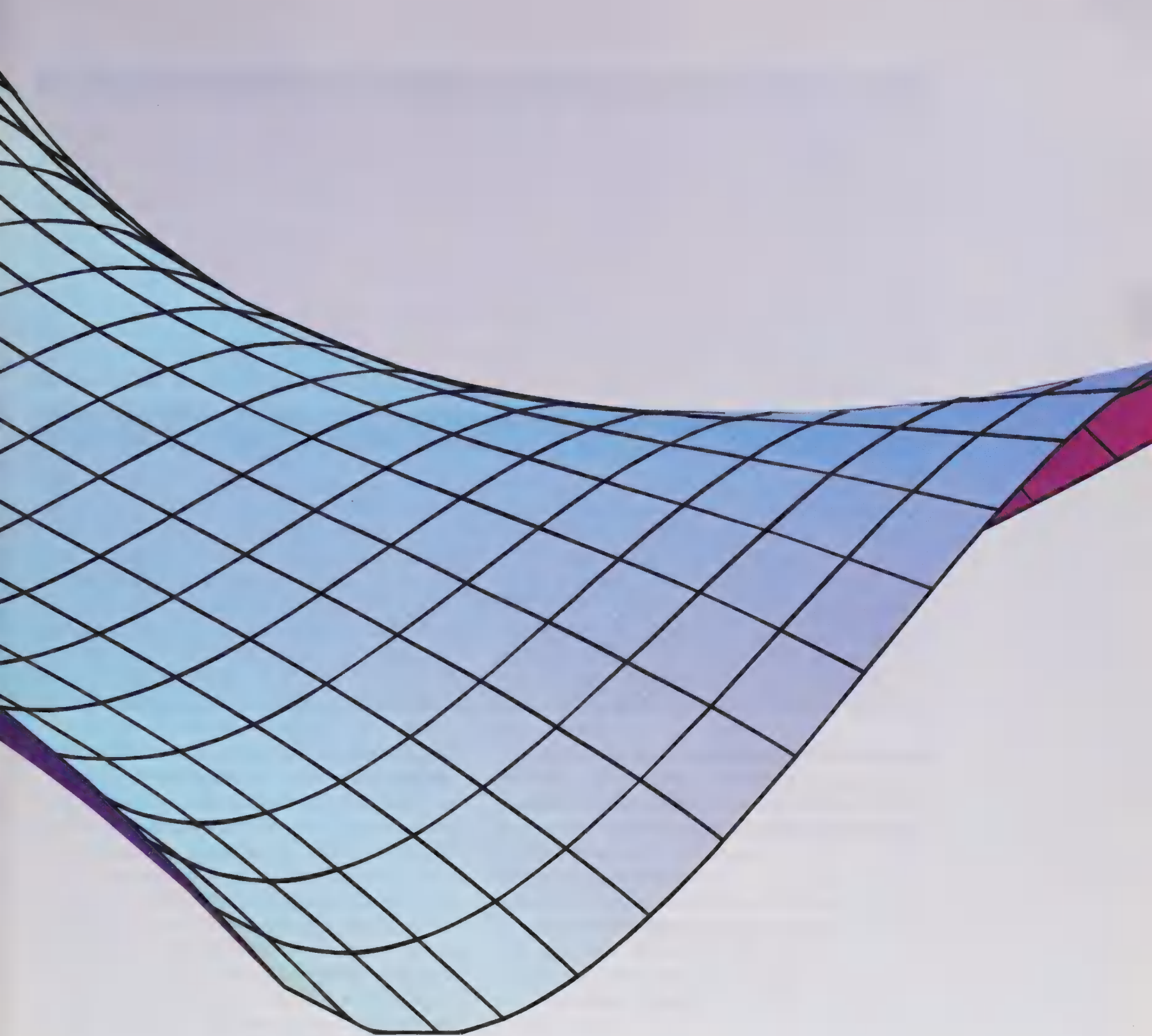
Much of what we teach, especially in college, is designed to support other disciplines. To determine the current mathematical needs of twenty-three partner disciplines, the Mathematical Association of America (MAA) conducted the Curriculum Foundations Project (Ganter and Barker 2004; Ganter and Haver 2011), as discussed in the appendix (www.nctm.org/mt042). Overall, the recommendations from the partner disciplines reflected an amazing degree of unanimity.

One surprising recommendation was to introduce functions of several variables into introductory mathematics courses. To most mathematicians, this concept probably suggests partial derivatives, gradi-

ents, and multiple integrals—topics not accessible before calculus. Physicists, economists, and chemists, for instance, have a different interpretation; they want the same multiple representations—symbolic, numerical, and graphical—that are the mainstays of algebra through calculus. In this article, we discuss how these multiple representations apply to functions of two variables in intermediate algebra or precalculus and provide examples to illustrate the request of the partner disciplines.

NUMERICAL REPRESENTATIONS

In most fields, we routinely encounter information through tables that often are sets of values depending on two distinct quantities. For instance, the



windchill factor C adjusts temperature readings T to take into account the wind speed W to measure how cold it feels. **Table 1** shows some values for C , with C and T in degrees Fahrenheit and W in miles per hour. Thus, if $T = 20^\circ$ and $W = 25$ mph, the windchill factor is $C = 3^\circ$.

Students in courses in many other fields are expected to read and interpret such tables and perhaps even create them. This process involves a considerable amount of mathematical understanding and reasoning and is not as simple for many students as we might anticipate.

We start with a simpler example (see **table 2**). If we carefully examine the values of z , we will notice that, along every row, each successive value of z is

3 more than the previous value. That is, for each value of y , the row is a linear function of x and the slope of that line, m_x , is 3. Similarly, down every column, each successive value is 30 more than the previous value. That is, for each value of x , every column is a linear function of y and the slope of that line is $m_y = 30/5 = 6$. Thus, z is a linear function of the two variables x and y , and the resulting equation is $z = 3x + 6y - 11$. The constant, -11 , is the value of z that corresponds to $x = y = 0$ and is the *vertical intercept*. More generally, the slope-intercept form for a linear function of two variables is $z(x, y) = m_x x + m_y y + c$.

The parallels with standard ideas about linear functions are very clear. Introducing functions of

Table 1 Some Windchill Factor Values (in °F)

Wind Speed (<i>W</i>) (in mph)	Temperature (<i>T</i>) (in °F)							
	35	30	25	20	15	10	5	0
5	31	25	19	13	7	1	-5	-11
10	27	21	15	9	3	-4	-10	-16
15	25	19	13	6	0	-7	-13	-19
20	24	17	11	4	-2	-9	-15	-22
25	23	16	9	3	-4	-11	-17	-24
30	22	15	8	1	-5	-12	-19	-26

Table 2 *z* as a Linear Function of *x* and *y*

		<i>x</i>						
<i>y</i>		0	1	2	3	4	5	6
	0	-11	-8	-5	-2	1	4	7
	5	19	22	25	28	31	34	37
	10	49	52	55	58	61	64	67
	15	79	82	85	88	91	94	97
	20	109	112	115	118	121	124	127

several variables is not just a matter of adding a new dimension to algebra or precalculus. It also provides an opportunity to reinforce fundamental concepts about linearity while introducing students to notions that are important in other subjects. For instance, students can be asked to find the equation of linear functions across different rows or down different columns, and the teacher can compile the results on the board. In the process, these ideas clearly connect to topics that students have seen in real life and that they can appreciate as being important.

In practice, few functions that model data found in tables are linear. If we examine the rows and columns in **table 1**, it is clear that neither data set is linear. (The actual formula for the windchill factor is $C = 35.74 + 0.6215T - 35.75W^{0.16} + 0.4275TW^{0.16}$, with T in °F and W in mph, although we would not recommend having students work with it.) Consequently, trying to predict windchill factors when values for T and W are not listed is challenging for many students. Whereas other disciplines did not mention interpolation, many emphasized that students need to work with proportions; certainly, mathematicians realize the importance of proportional reasoning. Thus, introducing problems involving interpolating in tables may be valuable.

To illustrate, let's estimate the windchill factor corresponding to $T = 18^\circ$ and $W = 23$ mph. We use the nearby point $T = 15^\circ$ and $W = 25$ mph,

where $C = -4^\circ$. In the direction of increasing T , the slope of the function is approximately $7/5 = 1.4$, so C increases by approximately 1.4° for every 1° increase in temperature. Therefore, corresponding to an increase of 3° , C changes by roughly $3(1.4) = 4.2^\circ$, with W a constant 25 mph. Similarly, in the W direction, when $T = 15^\circ$ and $W = 20$ mph, the slope is roughly $-1/5 = -0.2$, so C decreases by approximately 0.2° for every 1 mph increase in wind speed. Therefore, if W increases by 3 mph, C changes by roughly $3(-0.2) = -0.6^\circ$, with T a constant 15° . Consequently, the total change in C is approximately $4.2 - 0.6 = 3.6^\circ$, and we estimate that when $T = 18^\circ$ and $W = 23$ mph, $C = -4 + 4.2 - 0.6 = -0.4^\circ$. Is this result reasonable?

Most of the fundamental concepts of functions of one variable extend in natural ways to functions of two variables. The formal definition of a function $z = f(x, y)$ says that for every possible pair of x - and y -values, there corresponds precisely one value of the dependent variable z . In terms of the windchill factor, if a value for T and a value for W lead to two different values for C , then **table 1** would not satisfy the definition.

The notions of domain and range similarly extend to functions of two variables. The domain consists of all possible pairs of values (x, y) , and the range consists of all corresponding values of the dependent variable z that arise. For instance, with the windchill factor, the values shown for T extend

from 0° to 35° , and those shown for W extend from 5 to 30 mph. If these are the only possible values for T and W —that is, if the function cannot be extended beyond the values in the table—then we have the domain. **Table 1** shows that the corresponding values for the windchill factor C extend from -44° to 33° ; this would be the range.

Presumably, we could extend the values for T —say, from a minimum of -40° to a maximum of 45° (windchill likely would not make sense at warmer temperatures). W could be extended from 0 mph (we cannot have a negative wind speed) to a high of 60 mph. The resulting domain is T between -40° and 45° and W between 0 and 60 mph. The range would consist of all the associated values for the windchill factor. The full table of windchill temperatures is available on the National Weather Service website at <http://nws.noaa.gov/om/windchill/>.

We can also extend the fundamental notions about the behavior of functions to describe the behavior of functions of two variables. With a univariate function, recognizing increasing or decreasing behavior graphically or numerically is simple. But a multivariate function can increase as both independent variables increase; it can decrease as both variables increase; or it can increase as one variable increases and decrease as the other increases. Comparable issues arise with concavity.

With the windchill factor, as T rises and W remains fixed (looking horizontally from right to left along each line of the table), the values for C increase. Thus, C is an increasing function of T . Alternatively, as W rises and T remains fixed (looking down each column), C gets smaller and eventually more and more negative. Thus, C is a decreasing function of W .

Moreover, because C does not change linearly in either direction, questions of concavity come into play. When C increases, does it increase at an increasing rate or a decreasing rate? When it decreases, does it decrease at an increasing rate or a decreasing rate? Alternatively, in a somewhat more dynamic form: Does C increase faster at low temperatures or at high temperatures? Does C decrease faster at low wind speeds or at high wind speeds? Questions such as these make for good classroom discussions that engage students or for homework problems that require students to think carefully about the mathematical and practical issues.

GRAPHICAL REPRESENTATIONS

When mathematicians hear the words “graph of a function of two variables,” they invariably think of surface plots. In almost all other disciplines, the standard graphical depiction is a *contour diagram*, or *contour map*. **Figure 1** shows the contour diagram for the heights (in feet) of the terrain on a

hypothetical island. Each contour, or *level curve*, connects points that are all at the same height. The shore around the island is at sea level, so it is represented as the closed contour with height 0 feet. The other contours are all multiples of 500 feet. For instance, the 1000 contour represents all heights that are 1000 feet above sea level. As such, these contour curves form a series of closed loops.

Moreover, the contour map shows three distinct groups of closed loops, with each successive contour smaller in size and larger in value. Each of these nested loops represents a mountain. The peak toward the left is slightly higher than 3000 feet; the one to the right is somewhat higher than 2000 feet; and the one in the upper center is higher than 3500 feet.

Clearly, the level curves represent a sample of heights; it is not possible to give the height of every point. However, one of the “rules” for creating contour plots is that a uniform interval, or *contour interval*, exists between contour values. The contour interval in **figure 1** is 500 feet.

The absence of a contour also provides useful information. In **figure 1**, the highest contour is 3500 feet. If the height reached 4000 feet at any point, we would see a 4000-foot contour. Therefore, the maximum height must be between 3500 and 4000 feet.

Another rule for contour plots is that different contours cannot intersect. If they did, then there would be two different “heights” associated with a single point, a condition that violates the definition of a function. Realistically, such an intersection can happen in nature—if there is an overhang in the terrain, for instance.

Consider the contour diagram in **figure 2**, which shows heights in San Francisco, known for its hilly terrain. For reference, the horizontal distance between California Street and Clay Street is about 600 feet. Students could be asked, for instance, to estimate the height of Nob Hill, the height at the intersection of Clay and Grant Streets,

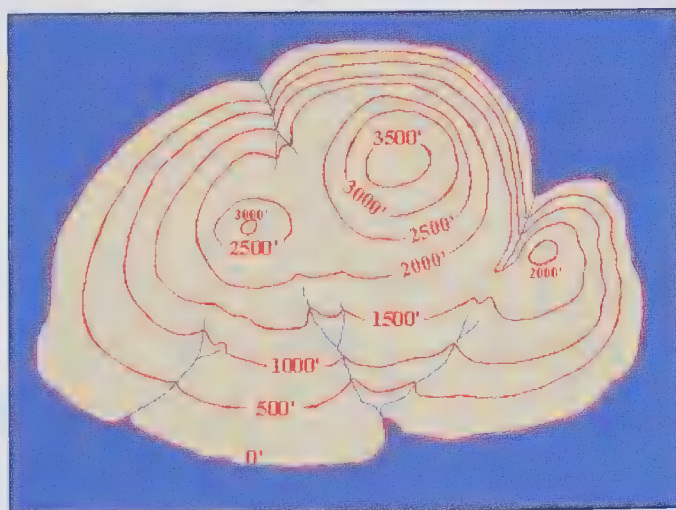


Fig. 1 This contour map indicates three mountains on the island.

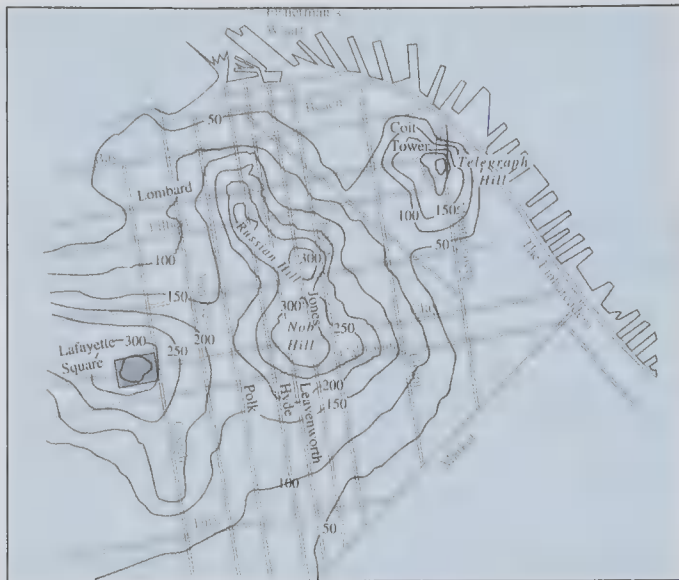


Fig. 2 The distance between California and Clay Streets in San Francisco is approximately 600 feet.

or the slope of the climb from Clay and Grant Streets to the top of Nob Hill.

We can extract considerably more information from a contour diagram than just the height at the points along the contours. The relative distances between adjacent contours tell us much about the behavior of the quantity being investigated (see **fig. 3**). Regions where the successive contours are far apart indicate that the function is relatively constant, so any changes are slow and, therefore, the slope is small. Alternatively, regions where successive contours are close together indicate that the function changes rapidly, so the slope is large.

Given the scale for a contour diagram, students can apply some simple ideas on slopes to estimate the slope of a terrain or the rate at which any quantity changes. All that is required is to select

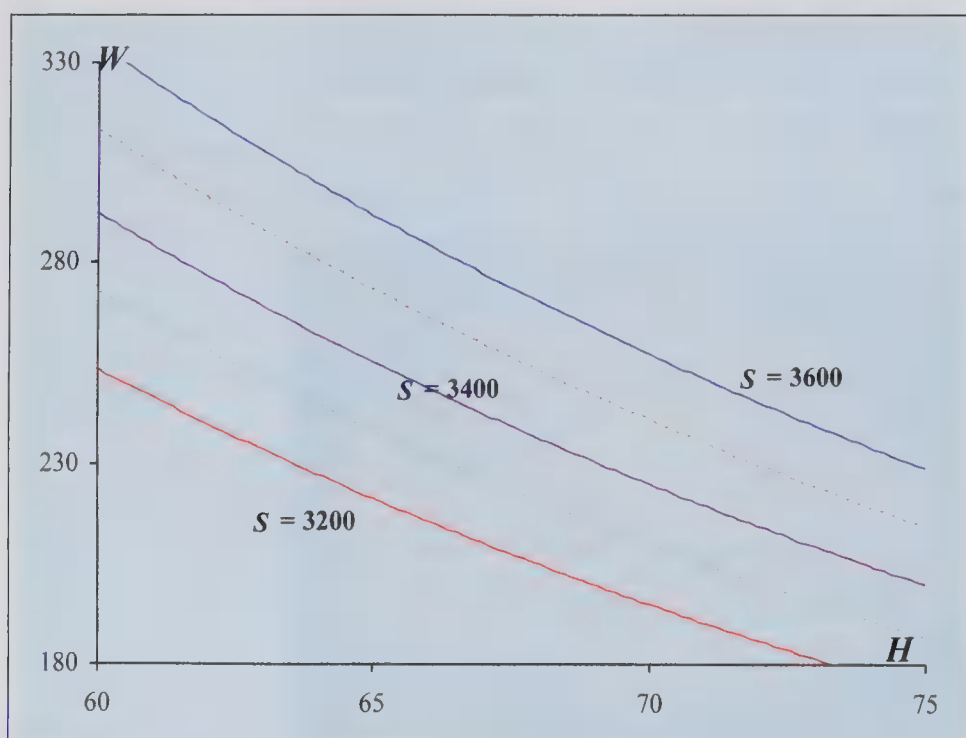


Fig. 3 The spacing between successive contours is sometimes difficult to discern.

points on two different contours (so that we know the change in “height”), measure the distance between the two points, and use the usual formula for the slope to estimate its value. As such, this is another way to reinforce earlier ideas on linear functions.

Students can engage in other, more exploratory activities. For instance, sediment from Catalina Island, off the coast of California, has created an underwater trench, the Catalina Canyon, along which silt flows to the sea floor. Students, either individually or in small groups, can be asked to locate the head of this “canyon” and draw a curve tracing the path of the silt flowing down the trench. Or they can be asked to identify areas that might be good for a beach or for snorkeling. Students can then be asked to use the Internet to verify whether a beach does exist near where they predicted. (For other examples and related classroom activities, see Fey and Heid [1995], Gordon and Gordon [2010], and Souhrada [2006].)

SYMBOLIC REPRESENTATIONS

When teachers from other disciplines talk about students needing to know about functions of several variables, they usually mean that students need facility working with certain basic formulas that represent important relationships among three (or more) quantities. In chemistry, a key principle is the ideal gas law, $PV = nRT$, which relates the pressure P of a gas, its volume V , and the temperature T ; n is the number of moles, and R is a constant. Chemistry teachers assign students problems in which two of the variables are given and students have to solve for the third.

A famous biological model relates a person’s surface area S (in in.^2) to his or her height H (in in.) and weight W (in lb.) through the formula $S = 15.64W^{0.425}H^{0.725}$. Some typical questions might be these: Estimate the surface area of a person whose height is 70 in. and whose weight is 180 lb., or estimate a person’s weight whose surface area is 3500 in.^2 and whose height is 65 in.

Similarly, one version of the Cobb-Douglas model in economics relates production P (the monetary value of all goods produced) to the labor input L and the capital input K through the formula $P = AL^\alpha K^{1-\alpha}$, $0 < \alpha < 1$. Economists expect students to solve for one of the variables given the other two.

These examples may not seem like deep mathematics, but they can be challenging to students in those other, nonmathematics courses, possibly because the variables are not x s and y s.

Another classroom suggestion is to have students change the units in the body-surface-area model from in.-lb.-in.^2 to m.-kg.-m^2 or cg.-kg.-m^2 to create comparable formulas.

Students in these other fields are also expected to interpret and use contour plots for similar functions. Although we addressed the issues involved in interpreting contour maps, we did not indicate how we would construct them according to a formula. This can be a classroom activity at the algebra or precalculus level that, again, provides the opportunity to remind students about the behavioral properties (increasing vs. decreasing, concavity) and algebraic properties of various families of functions.

A Classroom Activity

Suppose that the contour diagram for the human surface area function, $S = 15.64W^{0.425}H^{0.725}$, is to be plotted with H horizontally and W vertically. Have students answer the following questions:

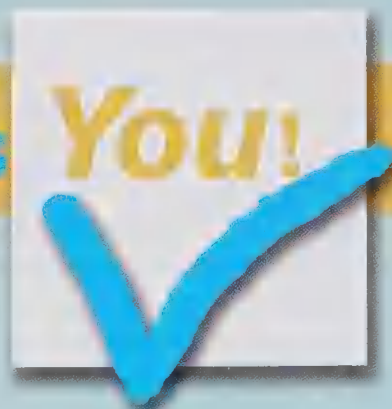
1. Determine the shape of the contours. The level curves associated with functions such as these are typically increasing and concave up, increasing and concave down, decreasing and concave up, or decreasing and concave down. Decide which possibility is correct. As a start, pick a value—say, $S = 3400 \text{ in.}^2$ —substitute it into the formula, and solve algebraically for W as a function of H . Then use knowledge of the behavior of power functions to deduce the shape of the contours.

2. Do successive contours get larger or smaller in a certain direction? That is, as you move upward and toward the right, for example, do the successive contours correspond to larger or smaller values for S ? Hint: Use the $S = 3400 \text{ in.}^2$ contour from part (1) as a baseline and decide whether the “next” contour in that direction is $S = 3300 \text{ in.}^2$ or $S = 3500 \text{ in.}^2$.

Alternatively, we can argue in a more dynamic though less precise way. What should happen to the surface area if both height and weight increase? Although not conclusive, a good model should reflect the underlying process, and students should learn to develop their mathematical intuition to understand and predict what will likely happen with a model with respect to reasoning and sense making.

3. Does the spacing between successive contours increase or decrease? Students can use the two contours from parts (1) and (2) along with a third contour—say, $S = 3200 \text{ in.}^2$ or $S = 3600 \text{ in.}^2$ —to decide whether the distances between the successive contours increase or decrease with help from a graphing calculator. A little effort to generate an appropriate window in which all three curves can be seen clearly may be required, and sometimes deciding whether the spacing is increasing or decreasing can be rather hard. Prediction is difficult

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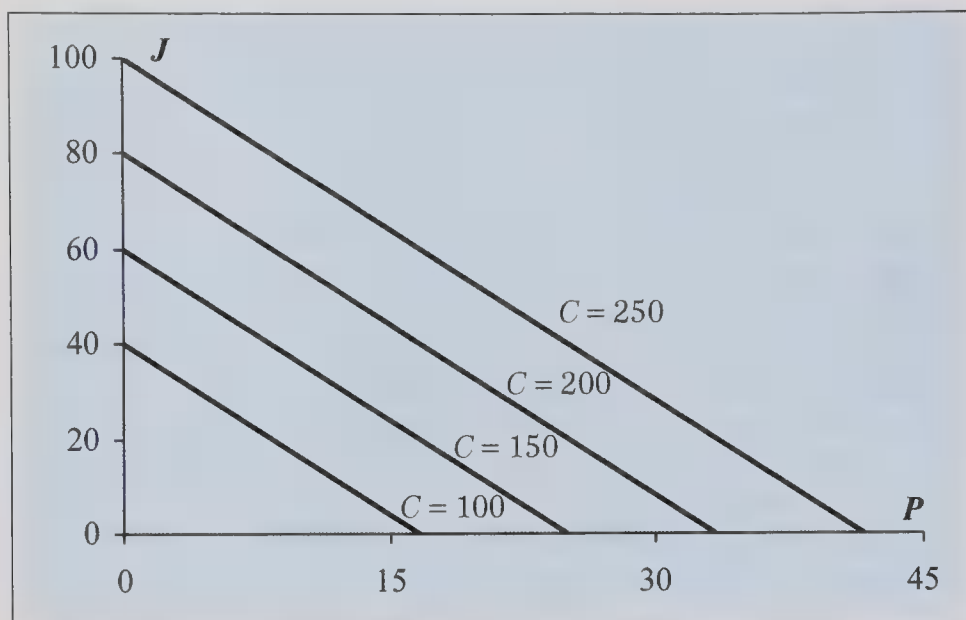


Fig. 4 Contour lines here are parallel and equally spaced, as one would expect.

in this case, as seen in **figure 3**, although the spacing appears to be increasing, albeit very slowly, as S increases from 3200 to 3600.

CONTOUR PLOTS OF LINEAR FUNCTIONS

What does the contour plot associated with a linear function $z = ax + by + c$ look like? There are several ways to proceed.

One approach is to use some algebra. Let's start with a simple application—the number of calories C in a peanut-butter-and-jelly sandwich. One brand of peanut butter contains 6 calories per gram and a brand of jelly contains 2.5 calories per gram. White bread contains 75 calories per slice, so the total number of calories is $C = 150 + 6P + 2.5J$. Students can be asked to consider the contours corresponding to, for example, $C = 100, 150, 200, \dots$ to produce the normal equations for a set of lines and observe that they are all parallel (the same slope) and appear to be equally spaced (see **fig. 4**).

Alternatively, we can use a simple equation—for instance, $z = 2x + 3y + 1$. Students can be asked to create a table such as **table 2** (although a larger table with more entries might be more effective), plot all the points on a sheet of graph paper along with the associated values of the function z , and then connect the points corresponding to the same values of z . (If the table is extended far enough, most of the values of z will occur at least three times, giving a much more dramatic impact.) The results will again be a series of parallel, equally spaced lines. A third somewhat more sophisticated but generalizable geometric argument can be found at www.nctm.org/mt04s2.

CONCLUSIONS

At the college level, the overwhelming majority of students who take introductory mathematics courses do so to fulfill requirements for other disciplines. This approach is quite different at the high

school level, where mathematics requirements are imposed districtwide or statewide. Nevertheless, offering courses that meet the needs of both students and colleagues in other disciplines is critical.

Such courses are particularly helpful if we can introduce such topics in ways that develop and reinforce fundamental mathematical topics in students' minds while extending these topics in new directions. From the perspective of teachers in other fields, such courses provide their students a more solid mathematical foundation that directly supports the work done in those other courses and, further, allows teachers to increase the level of mathematical sophistication. In turn, such courses can encourage many students to take more advanced mathematics electives because they see mathematics' value in other fields.

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SHELDON P. GORDON, gordonsp@farmingdale.edu, is a SUNY Distinguished Teaching Professor at Farmingdale State College in New York. He seeks to improve the learning experience for all students while trying to modernize the mathematics curriculum at all levels.

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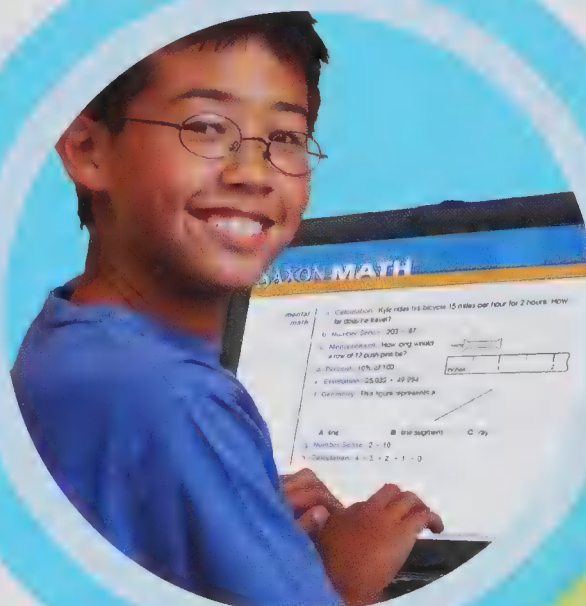
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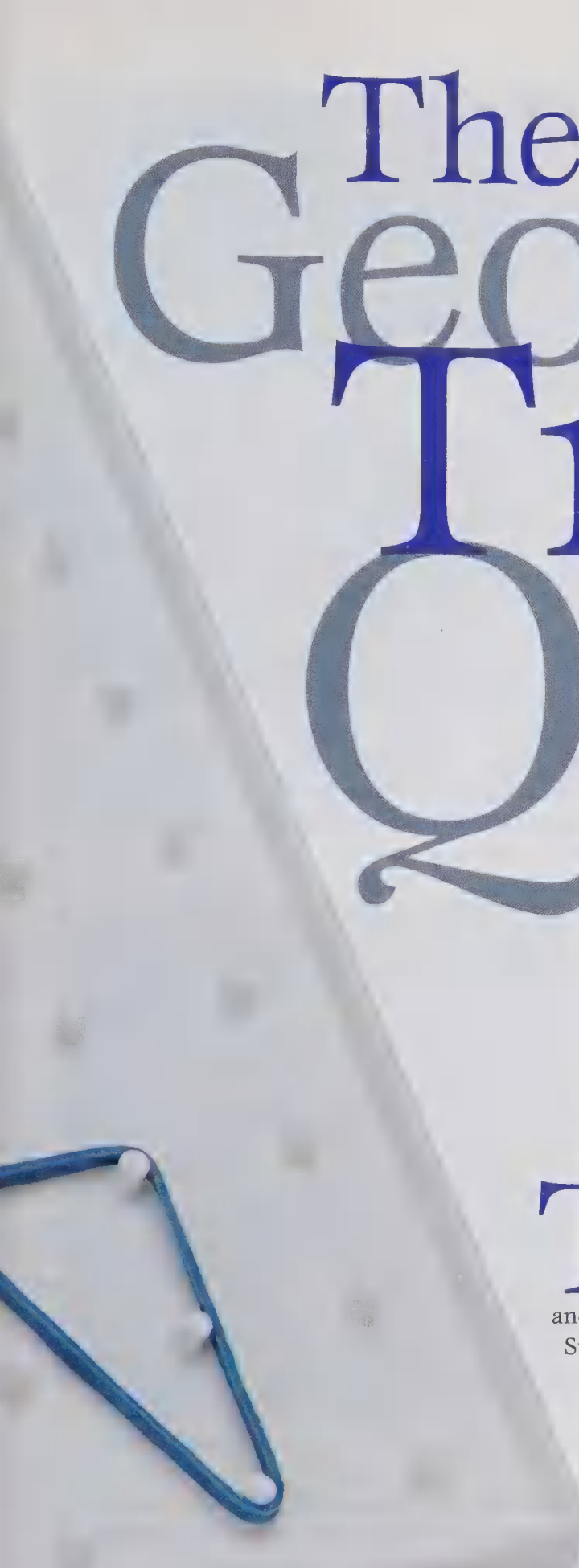
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A student question and the process it generates demonstrate the importance of student-initiated tasks in implementing the SMP in the Common Core Standards.

Kasi C. Allen

The Common Core State Standards challenge us to foster vibrant and interactive mathematics learning communities in our classrooms, communities in which the student-teacher hierarchy gives way to collective ownership of the mathematics as all work together to generate and refine mathematical ideas. Within the Common Core (CCSSI 2010), the Standards for Mathematical Practice (SMP) portray a vision of content-rich classes in which students are engaged in frequent problem-solving experiences that compel them to persevere, use a variety of tools, and develop powerful arguments that include not only their own ideas but also those of others—all with precision, strategic thinking, and growing awareness of the powerful structures underlying mathematics as a discipline (see **fig. 1**). Teachers who truly commit to creating such classrooms must acknowledge the value of empowering students to generate their own problems and prepare for the pedagogical adventure that ensues.

This is the story of such a student-initiated problem, a seemingly simple question posed by students in my geometry class years ago:

How many triangles can you make on a geoboard?

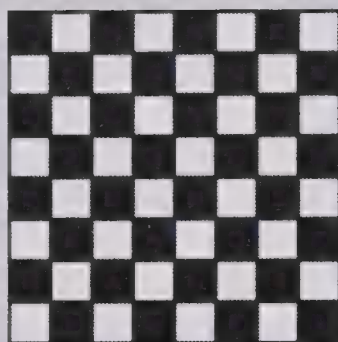
Facilitating the discussion and exploration that followed changed me as a teacher. Over the years, I have used the Geoboard Triangles problem with my students at various grade levels, from high school students to preservice teachers. The quest for geoboard

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Source: CCSSI (2010), p. 10

Fig. 1 The Standards for Mathematical Practice are designed to produce vibrant and interactive mathematics communities.



(a)

Square Type	Number of Squares
1×1	64
2×2	49
3×3	36
4×4	25
5×5	16
6×6	9
7×7	4
8×8	1

(b)

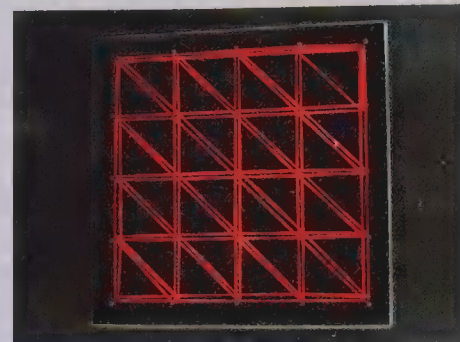
Fig. 2 In their pursuit to find all the squares “hidden” in the checkerboard, students use multiple strategies to generate lists and tables.

triangles never fails to provide adventure and truly reflects the instructional vision of the Common Core.

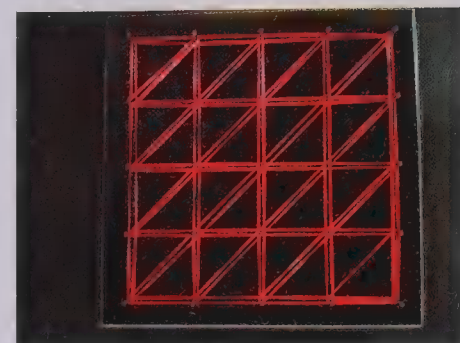
DEVELOPING THE PROBLEM

When my students first posed the Geoboard Triangles problem, our class had just finished solving the classic Checkerboard Squares problem: How many squares are in a checkerboard? (See **fig. 2**.)

In the whole-class debriefing, small groups shared their strategies and findings, noting that



(a)



(b)

Fig. 3 Students represent 64 isosceles right triangles with legs 1 unit long.

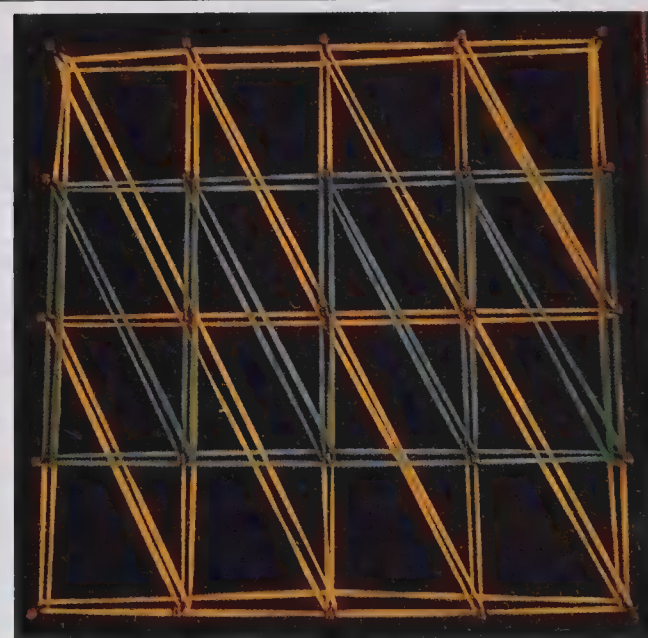


Fig. 4 Students experiment with 1×2 right triangles.

once they could locate all 49 of the 2×2 squares, they readily found the rest. Many students pointed out the pattern of perfect squares in their tables. When the class talked about possible extension questions, students in one small group asked: “What if we used geoboards? What if we wanted to figure out how many triangles are on a geoboard?” (A geoboard is a square pegboard with 25 pins arranged in a 5×5 array.) My students, convinced that they would uncover another “hidden” numeric pattern, headed to the shelves for geoboards and rubber bands.

Although arguably related, the Geoboard Triangles problem did not prove to be the extension of the Checkerboard Squares problem that my students had anticipated. It was far more complex than they had imagined. Issues of orientation, reflection, and rotation surfaced almost immediately, leading students to revise the original problem.

Students began their initial triangle search by considering the case of right triangles with two equal legs 1 unit long (see **fig. 3**). They were especially excited to find 64 of them; the perfect square contributed to their sense that the solution to this problem might involve a pattern similar to the one they had uncovered in the Checkerboard Squares problem.

However, as students explored right triangles with legs 1 and 2 units long, respectively, some began recognizing the problem’s inherent complications. They counted the yellow triangles (see **fig. 4**) and got 16. The reflection would give another 16. There were squeals of delight—“16 plus 16 is 32, and 32 is half of 64!” Then one student showed that a rotation of 90 degrees clockwise would give another 16 triangles and that a rotation of 90 degrees counterclockwise would give yet another 16. Soon afterward, another student placed the green rubber bands as shown in **figure 4**, yielding another group of 8 triangles that could be translated similarly. The predicted pattern disintegrated.

As students debated about what to do next, one group focused on isosceles right triangles and generated a table similar to **table 1**.

Another pair created the design in **figure 5**, announcing that these were the triangles that had a side length of $\sqrt{2}$. This discovery led to a discussion of “triangle types” and “triangle families” (students’ language). Students talked about how the isosceles right triangles behaved a little like the checkerboard squares. However, as a class, they seemed fairly awed by their growing awareness of just how many different triangles could be represented on a geoboard.

This line of thinking led students to reformulate their question—How many unique triangles can be made on a 5×5 geoboard?—and ultimately to the version of the Geoboard Triangles problem that I have offered my students ever since. Further classroom discussion clarified that what students meant by *unique* was *noncongruent*. I recommended replacing *made* with *represented*. The final reformulation then read:

How many noncongruent triangles can be represented on a 5×5 geoboard?

PURSuing A SOLUTION

Over the years, I have documented patterns in the strategies that students use when encountering

Table 1 Geoboard Isosceles Right Triangles		
Length of Leg	Length of Leg	Number of Triangles
1	1	64
2	2	36
3	3	16
4	4	4

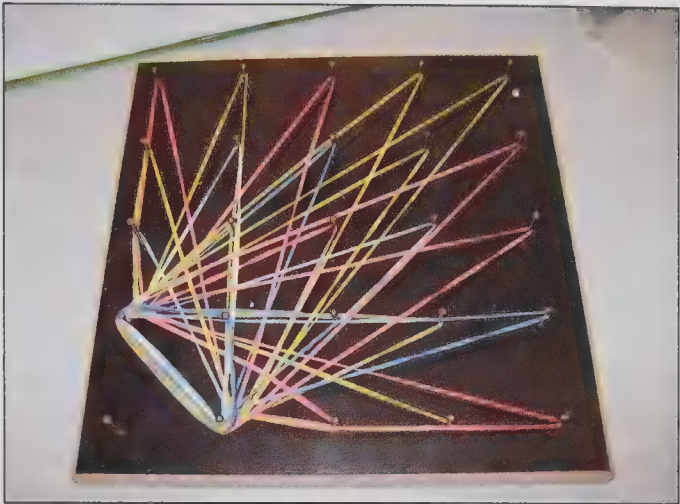


Fig. 5 One pair of students created triangles with side length equal to $\sqrt{2}$ units.

the Geoboard Triangles problem for the first time. However, time has changed the tools that students have at their disposal—from physical geoboards and dot paper sketches to a range of computer-based applications—making it possible for students to represent their ideas more readily, organize their work, and arrive at a solution that they can verify. These tools and sources include the following:

- “Making Triangles” applet (NCTM), available to NCTM members on its website: <http://www.nctm.org/standards/content.aspx?id=25007>
- Multiple geoboard applets offered by the National Library of Virtual Manipulatives: <http://nlvm.usu.edu/en/nav/vlibrary.html>
- Texas Instruments applet: <http://education.ti.com/us/product/apps/geoboard.html>
- Multiple sources for downloading 5×5 geoboard paper: see, e.g., <http://mathforum.org/trscavo/geoboards/dot1.html>

Beginning with Individual Exploration

Regardless of their age or experience, students benefit from some individual time for reflection before collaborating with their peers on a task this complex. When first considering the problem, students generally focus on the right triangles with integral side lengths.

Most students begin by counting the right triangles that have side lengths of 1, 2, 3, or 4 units; there are 10 of these. Students can confirm this result by creating a systematic list, and many will

Table 2 Right Triangles with Leg Lengths of 1, 2, 3, and 4 Units		
Leg 1	Leg 2	Hypotenuse
1	1	$\sqrt{2}$
1	2	$\sqrt{5}$
1	3	$\sqrt{10}$
1	4	$\sqrt{17}$
2	2	$\sqrt{8} = 2\sqrt{2}$
2	3	$\sqrt{13}$
2	4	$\sqrt{20} = 2\sqrt{5}$
3	3	$\sqrt{18} = 3\sqrt{2}$
3	4	5
4	4	$\sqrt{32} = 4\sqrt{2}$

do so without prompting. **Table 2** presents a list of these triangles, with the lengths of each hypotenuse, something that students generally do not include in their early stages of exploration. However, this information will prove helpful as they continue, so I often encourage them to record the lengths of all the sides.

Another initial student strategy involves counting triangles with integral-length base and height. Because some of these triangles are also right triangles, students who pursue this path immediately encounter issues related to eliminating repeated triangles, determining whether or not two triangles are congruent, and devising a counting system that will enable them to keep track of their findings as they continue their quest for geoboard triangles.

Figure 6 provides an example of two triangles that force the question of how best to name triangles. Both have a shortest side of 2 units (which also serves as the base), a height of 4 units, and an area of 4 units². However, students can see that the triangles are not congruent.

Fostering Collaboration

When small groups first convene following individual reflection time, I ask each student to bring to the group a conjecture and a question. Most students also bring with them geoboard diagrams. In the discussions that follow, I hear rich use of mathematical vocabulary, terms such as *reflection*, *rotation*, *right triangle*, *leg*, *hypotenuse*, *isosceles*, *congruent*, *area*, *base*, *height*, *square root*, *side length*, *parallel*, *points*, *distance*, and *eliminate*.

When we come together as a whole class, I ask the groups to share any new questions that need clarifying. Then I display an image similar to that in **figure 6** and ask the groups to talk for a few minutes about how to distinguish between the two

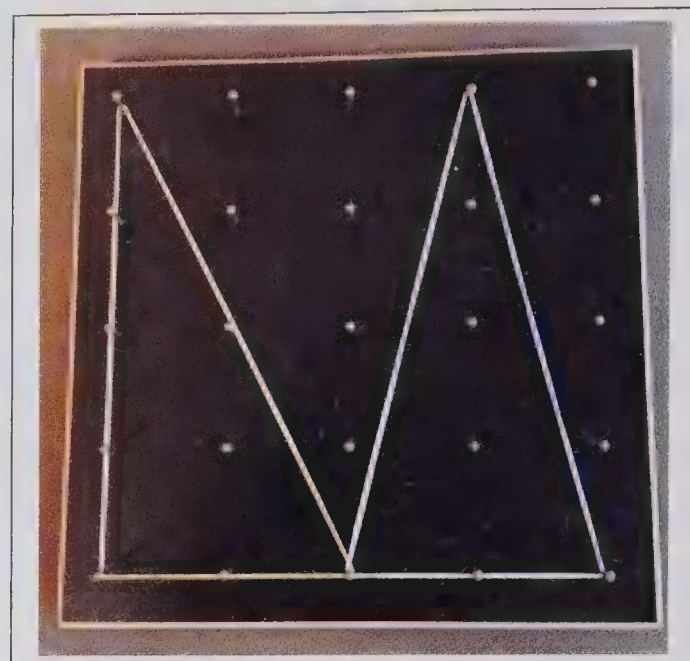


Fig. 6 Noncongruent triangles may have equal bases, heights, and areas.

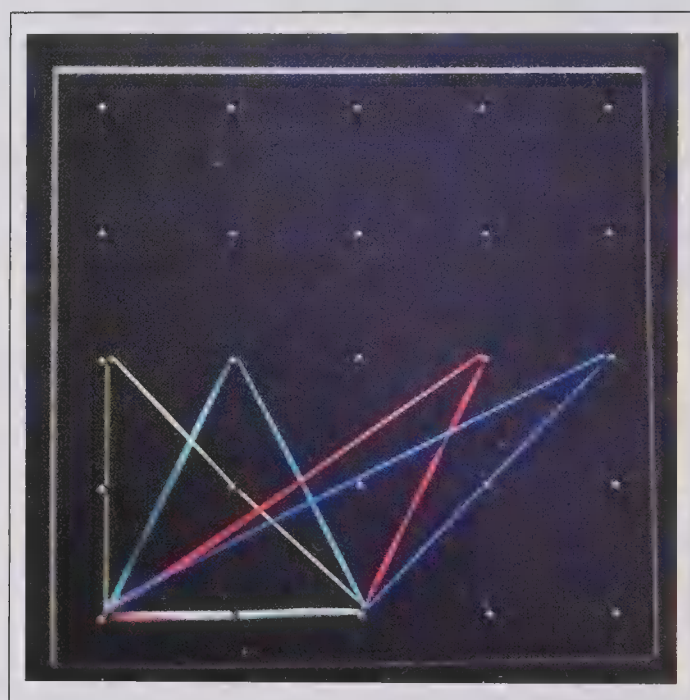


Fig. 7 Maintaining the same base of 2 units, students can construct four distinct triangles with a height of 2 units.

triangles shown. Next, I request that they articulate a strategy for keeping track of the triangles in their search.

In wrapping up these initial discussions, I generally ask each small group to write on an index card the basics of the group's strategy along with an estimate of the number of triangles that they think we will find. I post these index cards around the classroom for comparison, motivation, and revision in the days ahead.

REFINING THE SEARCH

As small groups proceed, students share representations and sort their lists, counting as they go. The challenge is twofold: (1) naming or labeling each triangle so that it can be identified as distinct from

Questions	Conjectures
▪ How many right triangles can you find with legs that have a whole-number length? ▪ How many isosceles right triangles can you find? ▪ Can you find isosceles triangles that are not right triangles? ▪ Can a triangle have a base whose length is a square root? ▪ How can you represent a length of 5 on the geoboard? ▪ What are all the possible side lengths that you can make? ▪ How can we distinguish triangles of equal area from one another?	• You cannot represent an equilateral triangle on a geoboard. • A reflection of a triangle is congruent to the original triangle. • A rotation of a triangle is congruent to the original triangle. • A translation of a triangle is congruent to the original triangle. • If two triangles have sides that are not equal, then the triangles are not congruent. • Two triangles can have the same area without being congruent. • It is possible to find a triangle whose sides are all irrational numbers.

Fig. 8 Geoboard questions and conjectures can come from both students and teacher.

the others; and (2) developing a tracking system to determine readily whether a seemingly new triangle is indeed unique or merely a repeat. Although students who initially use an area approach generally abandon it because of the problem of multiple noncongruent triangles with equal areas, groups that never fully engage in this discussion take longer to determine a system for organizing their work. **Figure 7** illustrates one of the most significant breakthroughs that students make with respect to keeping track of their findings: All four triangles represented have the same base and an area of 2 units. However, students can distinguish them by their side lengths, values that they calculate using the Pythagorean theorem.

Ultimately, students working to solve the Geoboard Triangles problem make use of the notion that three sides determine a triangle. In other words, if a student finds a triangle with a new combination of side lengths, then this triangle is not congruent to any of the others and should be added to the list. Essentially, this new triangle reflects the inverse of the triangle congruence postulate:

If the three sides of a triangle are congruent to the three sides of another triangle, then the two triangles are congruent.

Holding this discussion provides a great opportunity to infuse the lesson with a little mathematical logic.

Keeping Track

As students proceed with their quest, some will choose to work more visually, creating a series of geoboard diagrams and labeling them with key attributes as well as triangle count. For example, **figure 7** might be described as “all triangles with

base of 2 and height of 2; count is 4.” I have had students create lovely posters presenting solutions based solely on visual representations. More often, groups use carefully crafted numeric lists to keep track of the side length combinations. Given the four triangles in **figure 7**, they would record each of their three side lengths: $(2, 2, 2\sqrt{2})$, $(2, \sqrt{5}, \sqrt{5})$, $(2, \sqrt{5}, \sqrt{13})$, and $(2, 2\sqrt{2}, 2\sqrt{5})$. As it turns out, every side length involved in the solution to the Geoboard Triangles problem appears in the list of what students often refer to as the “easy triangles” (the right triangles with integral-length legs) made at the outset, provided that they calculate the length of the hypotenuse (see **table 2**).

Over the years, I have also had a few students take a more algebraic approach to their search by viewing the geoboard as the first quadrant in the coordinate plane, with the bottom-left corner labeled $(0, 0)$, the bottom-right corner labeled $(4, 0)$, the top-right corner labeled $(4, 4)$, and the top-left corner labeled $(0, 4)$. Employing coordinate geometry, they use the three vertices of any two triangles along with the distance formula to determine whether or not the triangles are congruent. Students with some knowledge of programming (either with a computer or graphing calculator) tend to be the ones who choose this path, attracted to the idea that they can systematically and mathematically test every combination of three points.

The Role of Questions and Conjectures

Regardless of the strategies they choose, students working to solve the Geoboard Triangles problem benefit from a classroom culture in which all feel comfortable generating conjectures and questions. In my own classroom, students may display these if they so choose. The practice serves to remind students constantly of the diversity of ideas being

discussed in the room and to broaden their own thinking, especially when they find themselves losing momentum or getting stuck. As the teacher, I also judiciously participate in posing questions and conjectures according to what I am seeing as I interact with groups in the classroom. **Figure 8** gives a sample of the sorts of questions and conjectures that surface.

The vast majority of students, with the support of their small group, are able to find nearly all the geoboard triangles. To my way of thinking, the problem should be much more about their process than their answer. Still, some students and groups will need a nudge now and again. When this happens, I recommend asking students to—

- explain their search strategy and share the number of triangles found thus far;
- articulate their system for recording triangles, noting any difficulties that they are having;
- represent on the geoboard a triangle that they have not yet counted and determine how this triangle is different as well as whether there are more triangles like it;
- fix a side length (such as $\sqrt{2}$ in **fig. 5**) and represent all triangles with this side; or
- change the orientation of the geoboard and view it as a diamond to see whether new triangles become visible.

More often than not, students simply need support in systematizing their work.

LESSONS LEARNED

What I value most about the Geoboard Triangles problem is that the quest began with a student question. This personal investment motivates students to learn and persist in mathematics (Middleton and Jansen 2011). Hearing the history of the Geoboard Triangles problem inspires them to persevere as well, even when the process seems daunting or tedious. As they devise plans for finding a solution and implement them, often revising as they go, students experience the thrill and empowerment associated with creating mathematics. These, in turn, foster a desire to generate more questions of their own.

The Geoboard Triangles problem presents what Willis refers to as an “achievable challenge,” something that requires students “to exert mental effort, performing a task that is just difficult enough to hold their interest but not so difficult that they give up in frustration” (Willis 2010, pp. 17). From the standpoint of equity, the search element inherent in the Geoboard Triangles problem supports multiple access points, making it possible for each student to contribute to the solution in some way.

The rush of the school year and the pressure of test scores seldom support opportunity for teenagers to engage with tasks such as the Geoboard Triangles problem. We must make time—time for students to explore, time to make sense of what they discover, and time to give them choices about how they will proceed along their path of exploration. Problems like Geoboard Triangles can also help preserve time and relieve pressure by promoting mathematical tenacity and attending to multiple standards simultaneously.

According to the Common Core, eighth-grade students should “understand congruence and similarity using physical models, transparencies, or geometry software” and “understand and apply the Pythagorean Theorem” (CCSSI 2010, p. 53). High school students should “experiment with transformations in the plane” and “understand congruence in terms of rigid motion” (CCSSI 2010, p. 75). In short, the Geoboard Triangles problem is a simple case of less is more—a single problem with far-reaching benefits.

For those interested in the answer to the Geoboard Triangles problem, my students and I have concluded that there are 72 triangles.

REFERENCES

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- Willis, Judy. 2010. *Learning to Love Math: Teaching Strategies That Change Student Attitudes and Get Results*. Alexandria, VA: Association for Supervision and Curriculum Development.



KASI C. ALLEN, kasi@lclark.edu, is assistant professor of mathematics education in the Graduate School of Education and Counseling at Lewis and Clark College in Portland, Oregon. She is interested in issues of mathematics equity and strategies for mathematically empowering students across the grade levels.

Editor’s note: For more on counting polygons, see “Exploring Quadrilaterals through Flexible Approaches,” by Kerri Richardson, Anne Reynolds, and Catherine S. Schwartz (*MT* Nov. 2012, vol. 106, no. 4, pp. 288–94).



2015 Focus Issue

Creating Classroom Communities

NCTM guides mathematics teachers toward equitable teaching by emphasizing the importance of classroom communities. The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, and education researchers to share their experiences in building classroom communities. We encourage submissions that will help *MT* readers learn new ways to capitalize on the strengths that cultural, racial, and linguistic diversity bring to our classrooms and schools.

Classroom communities embrace individuals.

- How can we foster a sense of belonging in our classrooms?
- How can we learn about our students—their interests, issues that might be important to them, their languages, and their racial or ethnic communities?
- How can we incorporate this knowledge in our lessons and assessments? What are examples of effective tasks that highlight strengths of individual students? How can we balance individual, cooperative, and whole-class activities?

Classroom communities foster communication.

- How do we organize lessons so that students can share their mathematical ideas or solutions?
- What classroom norms are effective in facilitating communication?
- How can we encourage students to listen to, critique, and build on other students' mathematical thinking?
- How can we communicate high standards for students?
- How can technologies foster student communication about mathematics?
- What strategies are successful in removing barriers to student participation and engagement in mathematics?

Please submit manuscripts at mt.msubmit.net by **May 1, 2014**. Be sure to enter the call's title (Creating Classroom Communities) in the Department/Calls field. No author identification should appear in the text of the manuscript. See www.nctm.org/publications/content.aspx?id=22602 for manuscript guidelines. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.



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CALL FOR MANUSCRIPTS

Have you ever noticed that if you cancel the 9s in the fraction 19/95, you get the correct reduction, 1/5? Three other fractions consisting of two-digit numerators and denominators share this property. Can you find them? Do not count reciprocals or numerators and denominators that share a factor of 11.

1

Mars takes 686.971 Earth days to travel around the sun. If the months on Mars are proportional to the months on Earth, how many Earth seconds will there be in the month of September on Mars? (An Earth year is 365.25 days long.)

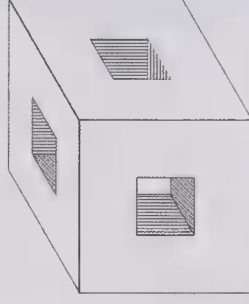
5

A circle is inscribed in a quarter circle such that the smaller circle is tangent to the two radii and to the arc of the quarter circle. Find the ratio of the area of the quarter circle to the area of the smaller circle.

9

What is the latest possible month of the year that the first Friday the 13th can occur? What year and month have the most Friday the 13ths? (Hint: Represent the seven weekdays as vertices of a polygon and the passage of time as motion around the polygon.)

A cube of edge 3 in. has 1-in.-square holes cut from three of its faces through to the opposite face. Find the area of the surface of the resulting shape, including its interior.



2

The domain of the function

$$f(x) = \sqrt{6 - \sqrt{2x + 7}}$$

is $\{x : a \leq x \leq b\}$. Determine

$$\frac{a+b}{2}.$$

6

The 25-question AMC exam is graded as follows: 6 points for a correct answer, 1.5 points for leaving the answer blank, and 0 points for an incorrect answer. A score of 100 or better is required to qualify for the AIME. If Sunil gets exactly 3 answers wrong, what is the least score that he can make to qualify for the AIME?

10

One factor of

$$3x^2 + bx - 20$$

is

$$x - 4.$$

Find b .

Two four-digit numbers, $abcd$, exist such that

$$abcd + a + b + c + d = 2013.$$

Find the two numbers.

3

The sum of two positive numbers equals the difference of the squares of the two numbers, which equals the quotient of the larger number when divided by the smaller. Find the smaller number.

7

The opening enrollment at Piedmont High School increased by 48 students in the first week of September. In the second week, the total population decreased by 5%. If 1216 students were enrolled at the end of the second week, what was the opening-day population?

11

Two concentric circles have radii a and b with $a < b$. A line tangent to the inner circle at P intersects the bigger circle at points Q and R . If $QR = 20$, find the area of the region between the two circles.

The product of

$$\sqrt[5]{8} \cdot \sqrt[3]{16}$$

can be expressed as 2^n . Find n .

4

The median of the set

$$\{23, 34, x, 49, 57, 90\}$$

is 46. Determine the value of x .

8

The length of the hypotenuse of a right triangle is 145. If the difference between the lengths of the legs is 119, find the sum of the two legs.

The three sides of a triangle are all integers. If two of the sides are 5 and 12, what is the ratio of the number of possible obtuse triangles to the total number of possible triangles?

16

Form all three-element subsets of the set of integers from 1 to 10, inclusive. Then find the product of the elements in each subset. How many of these products are divisible by 5?

20

Isosceles triangle ABC has vertex $\angle A$ and altitude AD . Extend ray DA through point A to a point P such that $PC = BC$. Locate a point E , not on $\overline{PC}$, such that $\angle PCA = \angle ACE$ and $CE = BC$. Find $m\angle AEC$.

24

Refer to problem 27. Explain how you can find the weight of each of the remaining objects in two more weighings.

28

Determine the largest integer that will divide

$$x^2(x^2 - 1)(x^2 - 4)$$

for all integral values of x .

17

How many seven-digit numbers of the form

$$a34567b$$

are divisible by 11, where a and b represent single digits?

21

The unit fraction $1/2$ can be written as the sum of two unit fractions:

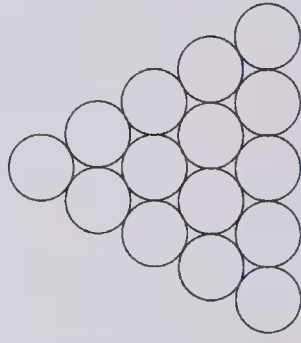
$$\frac{1}{3} + \frac{1}{6}$$

Find three unit fractions, all different, whose sum is $1/3$.

25

Place the integers 1 to 15 into the fifteen circles so that the sum of the five numbers

along each side of the triangle and the sum of the three central numbers will each be 39.



29

The sum of the squares of two numbers is 247, and the product of the two numbers is 21. Find all possible values for the sum of the two numbers.

18

In 1742, Christian Goldbach conjectured that every even integer greater than 2 can be written as the sum of two prime integers. Since then, no one has been able to prove or disprove the conjecture. To celebrate this fact, find the two primes that sum to 1742 and whose difference is the smallest.

22

A simple code assigns a number to each letter of the alphabet: 1 for A, 2 for B, 3 for C, and so on, to 26 for Z. Using these letters, decode this message:

201541259192082118194125

26

Altitudes drawn to the shortest two sides of a triangle have lengths 6 and 10. The length of the third altitude is greater than h . Find the minimum value of h .

30

Solve for all real values:

$$x \cdot x^{10/x} = \frac{x^x}{x^2}$$

19

A triangle is drawn with one side of length $20\sqrt{3}$; the altitude to that side is 4. If the measure of one angle adjacent to the given side is 30° , find the sum of the lengths of the other two sides of the triangle.

23

Four identical-looking objects weigh 3, 5, 8, and 11 oz., respectively. Using a balance scale, how can you determine the heaviest object in just two weighings?

27

Looking for more Calendar problems?

Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher*.

Edited by **Margaret Coffey**, Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and **Art Kalish**, artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791

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1. $16/64 = 1/4$; $26/65 = 2/5$; and $49/98 = 4/8 = 1/2$. Solving this problem requires a trial-and-success procedure, but attempting to solve it using a spreadsheet program such as Excel is also a great challenge. An Excel spreadsheet is available at www.nctm.org/mt. The spreadsheet assumes that the fraction $(10t + u)/(10u + v) = t/v$. If so, then we have the following:

$$\begin{aligned}\frac{10t + u}{10u + v} &= \frac{t}{v} \rightarrow \\ 10tv + uv &= 10ut + tv \rightarrow \\ 9tv + uv &= 10ut \rightarrow \\ v &= \frac{10ut}{9t + u}\end{aligned}$$

The spreadsheet can be set up so that t and u take on all possible single-digit values, generating values for v . When v is integral, a possible solution exists.

2. 72 in.^2 . Each face is missing a $1 \times 1 \text{ in.}$ square, so the area of the remaining portion is $9 - 1 = 8 \text{ in.}^2$. Removing the cube from a face adds 4 in.^2 of area. Therefore, for each face of the original cube, the area is 12 in.^2 . There are 6 such faces, so the total surface area is 72 in.^2 .

3. 2010 and 1992. Since $abcd + a + b + c + d = 2013$, $1001a + 101b + 11c + 2d = 2013$. Therefore, $a = 1$ or $a = 2$. When $a = 1$, we get $101b + 11c + 2d = 1012$. Thus, b must be 9, since the "carry" from the tens column cannot be more than 1. Now we have $11c + 2d = 103$. We know that c must be odd and big enough to carry 1 into the hundreds column, so $c = 9$. Then $d = 2$, and we are done with that possibility. When $a = 2$, $101b + 11c + 2d = 11$, so $b = 0$, $c = 1$, and $d = 0$.

We can prove that there are no other such numbers. Each time the units digit

increases by 1, the sum of the number and its digits increases by 2. For example, $1993 + 1 + 9 + 9 + 3 = 2015$. Each time the tens digit increases by 1, the sum of the number and its digits increases by 11. The maximum sum of four digits is 36, so the four-digit number must be greater than $2013 - 36 = 1977$. Since $1977 + 1 + 9 + 7 + 7 = 2001$, we wish to increase the units digit by 6 to increase the total by 12. Obviously, we cannot do so without affecting the tens digit. Increasing the tens digit by 1 increases the sum by 11, resulting in 2012. Increasing the tens digit by 2 increases the sum by 22, resulting in 2023. Thus, we then have to reduce the units digit by 5 to decrease the total by 10 and give us the result 2013.

4. $29/15$. Convert each radicand to a power of 2 and each radical to a fractional exponent. Therefore, $\sqrt[5]{8} = 2^{3/5}$ and $\sqrt[3]{16} = 2^{4/3}$. Multiply these numbers by adding the exponents: $3/5 + 4/3 = 29/15$.

5. 4,875,096 sec. An Earth year contains 365.25 days, so

$$\frac{686.971}{365.25} \cdot \frac{30 \text{ days}}{\text{mo.}} \cdot \frac{24 \text{ hr.}}{\text{days}} \cdot \frac{60 \text{ min.}}{\text{hr.}} \cdot \frac{60 \text{ sec.}}{\text{min.}},$$

or 4,875,096, is the number of Earth seconds in September on Mars.

6. $11/2$. Two restrictions need to be satisfied by the domain: $2x + 7 \geq 0$ and $6 - \sqrt{2x + 7} \geq 0$. The first is satisfied by $x \geq -7/2$. To satisfy the second, solve $-\sqrt{2x + 7} \geq -6$. Thus, $\sqrt{2x + 7} \leq 6 \rightarrow 2x + 7 \leq 36 \rightarrow 2x \leq 29 \rightarrow x \leq 29/2$. The domain of the function must satisfy both sets of conditions. The domain is $\{x: -7/2 \leq x \leq 29/2\}$, and the average of the extremes is $11/2$.

7. $\sqrt{2}/2$. Let $a > b > 0$. Then $a + b = a^2 - b^2 = a/b$. Then $a + b = (a + b)(a - b)$, so $a - b = 1$ and $a = 1 + b$. Substitute this into the equation: $a + b = a/b \rightarrow (1 + b) + b = (1 + b)/b \rightarrow b(1 + 2b) = 1 + b \rightarrow 2b^2 = 1 \rightarrow b = \sqrt{1/2} = \sqrt{2}/2$.

8. 43. The set has an even number of elements, so the median is the average of the two middlemost numbers. In this case, $(x + 49)/2 = 46 \rightarrow x = 43$.

9. $\frac{3+2\sqrt{2}}{4}$. Let the radius of the quarter circle = 1, and let the radius of the small circle = r . We can find r by applying the Pythagorean theorem or by knowing the ratio of the sides in a right isosceles triangle. Then, $r\sqrt{2} = 1 - r$, so

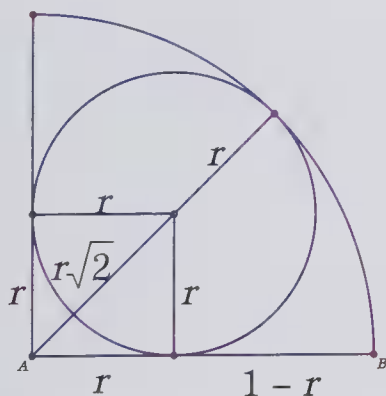
$$r(\sqrt{2} + 1) = 1 \rightarrow r = \frac{1}{\sqrt{2} + 1}.$$

We know that the ratio of the areas of similar figures equals the square of the ratio of corresponding linear measurements. Thus, the ratio of the area of the entire large circle to the area of the smaller circle equals

$$\left(\frac{1}{\frac{1}{\sqrt{2}+1}}\right)^2 = (\sqrt{2} + 1)^2 = 3 + 2\sqrt{2}.$$

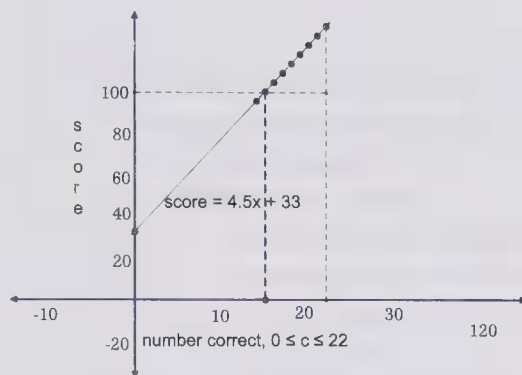
Since the requested ratio involves only the quarter circle, our final result is

$$\frac{3+2\sqrt{2}}{4}.$$



10. 100.5. If Sunil gets 3 answers wrong, 15 answers correct, and leaves the rest of the answers blank, he receives $15(6) + 3(0) + 7(1.5) = 90 + 10.5 = 100.5$ points. If he gets only 14 answers correct and leaves 8 blank, his score would be 96.

One way to approach this problem is to assume that Sunil answers the remaining 22 answers correctly, for a score of 132. For each question that he omits, he loses 4.5 points, so if we let x represent the number of omitted questions, we solve the inequality $132 - 4.5x \geq 100 \rightarrow x \leq 7.1$. Because x must be an integer, the number of omitted questions is 7, the number of correct answers is 15, and Sunil's score is $15(6) + 7(1.5)$.



Alternate solution: Find the score if 22 questions are answered correctly and if 22 answers are left blank. Plot the ordered pairs (c, s) — $(0, 33)$ and $(22, 132)$ —on the coordinate plane and connect them. Observe values of c where the segment intersects the line $s = 100$ and find $\lceil c \rceil$, the smallest integer greater than c .

11. 1232. Let x be the number of students on opening day. Then $(x + 48)(0.95) = 1216 \rightarrow x = 1232$.

12. 167. Let the legs of the triangle be x and y , so $x - y = 119$. Then $(x - y)^2 = x^2 - 2xy + y^2 = 119^2 = 14,161$. By the Pythagorean theorem, $x^2 + y^2 = 145^2 = 21,025$. Combining these two equations yields $2xy = 21,025 - 14,161 = 6,864$. Since $(x + y)^2 = x^2 + 2xy + y^2 = 21,025 + 6,864 = 27,889$, then $x + y = 167$.

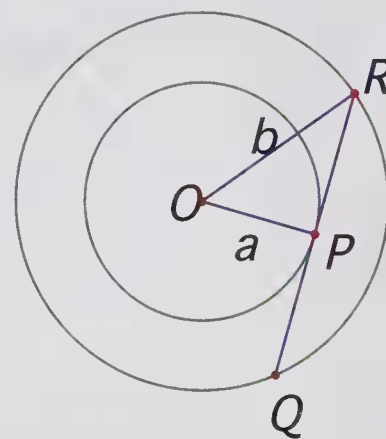
Alternate solution 1: Recognize that $(x + y)^2 = (x - y)^2 + 4xy = 119^2 + 2(145^2 - 119^2) = 2(145)^2 - 119^2$.

Alternate solution 2: In the solutions above, we obtained the requested sum without solving explicitly for the two legs. If x is the shorter leg, then $(x + 119)$ is the longer leg, and we have $x^2 + (x + 119)^2 = 145^2 \rightarrow 2x^2 + 238x - 6864 = 0 \rightarrow (x + 143)(x - 24) = 0$. The shorter leg is 24, the longer leg is 143, and their sum is 167.

13. Send student solutions to the Problem of the Month editors, Séan Madden (smadden@greeleyschools.org) or Ricardo Diaz (Ricardo.Diaz@unco.edu).

14. -7 . One factor is $x - 4$, so, for the first and last terms to be correct, the other factor must be $3x + 5$. The middle term is the sum of $-4(3x) + 5(x) = -7x$, so $b = -7$.

15. 100π . Point P is the midpoint of the tangent segment, so $PR = 10$. We wish to find the area of the region between the two circles: $\pi b^2 - \pi a^2 = \pi(b^2 - a^2)$. Since $OP \perp RQ$, $\angle OPR$ is a right angle, and we can use the Pythagorean theorem to obtain $a^2 + 10^2 = b^2$. Thus, $b^2 - a^2 = 100$, and the area of the ring is 100π .



16. $2/3$. According to the triangle inequality theorem, the only possible lengths for the third side are 8, 9, 10, 11, 12, 13, 14, 15, and 16. When the third side is 13 or $\sqrt{119} \approx 10.9$, the triangle will be a right triangle. Therefore, for all values greater than 13 or less than 10.9, the triangle becomes obtuse. There are six such values, so $6/9 = 2/3$.

17. 360. Factor to see that the structure of the expression is five consecutive integers: $(x - 2) \cdot (x - 1) \cdot x \cdot (x + 1) \cdot (x + 2)$ and an extra x . Five consecutive integers are always divisible by 5, and four consecutive integers are always divisible by 8, so we know that the five consecutive integers will be divisible by 40. We can regroup the six integers into two sets of three consecutive integers: $(x - 2) \cdot (x - 1) \cdot x$ and $x \cdot (x + 1) \cdot (x + 2)$. Three consecutive integers are divisible by 3, so the number is divisible by 9. Thus, the original expression will be divisible by $40 \cdot 9 = 360$.

18. ± 17 . Let the numbers be x and y . Then $x^2 + y^2 = 247$, and $xy = 21$. We wish to find $x + y$. Since $(x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 2xy = 247 + 2(21) = 289$, then $x + y = \pm\sqrt{289} = \pm 17$.

19. $-2, -1, 1, 5$. Apply the rules of exponents to rewrite the equation as

$$x^{10/x+1} = x^{x-2}.$$

The solution cannot be $x = 0$ given the domain of the original equation. When $x = \pm 1$, the original equation is true. The remaining solutions can be found by solving the equation $10/x + 1 = x - 2 \rightarrow 10 + x = x^2 - 2x \rightarrow x^2 - 3x - 10 = 0 \rightarrow (x - 5)(x + 2) = 0 \rightarrow x = 5$ or -2 .

20. 64. If the product of the elements is divisible by 5, then the subset must contain 5, 10, or both 5 and 10. If 5 is in the subset and 10 is not, ${}_8C_2 = 28$ subsets are possible. Similarly, if 10 is in the subset and 5 is not, ${}_8C_2$ subsets are possible. When both 5 and 10 are in the subset, there are 8 possible choices for the third element. Thus, $28 + 28 + 8 = 64$ subsets.

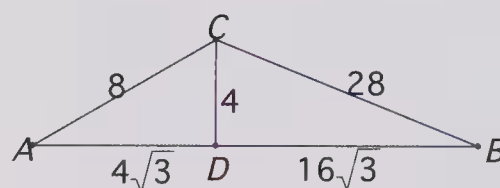
Alternate solution: From the total number of subsets, ${}_{10}C_3 = 120$, subtract the number of subsets not containing a 5 or 10: ${}_8C_3 = 56$. The result is $120 - 56 = 64$.

21. 8. For an integer to be divisible by 11, the sum of the digits in the even positions minus the sum of the digits in the odd positions must be a multiple of 11. For example, the number 25,839 is divisible by 11 since $(2 + 8 + 9) - (5 + 3) = 11$, which is a multiple of 11. Therefore, $(a + 4 + 6 + b) - (3 + 5 + 7) = 11n$ for some integer $n \rightarrow a + b - 5 = 11n$. Since a and b are single digits, the maximum value for the left side of the equation is 13, and the minimum value is -5 . Thus, when $n = 1$, $a + b = 16$, and we get three solutions: (9, 7), (8, 8), and (7, 9). When $n = 0$, $a + b = 5$, and we get five additional solutions: (5, 0), (4, 1), (3, 2), (2, 3), and (1, 4). Notice that (0, 5) is not a solution because a seven-digit number cannot begin with 0.

22. 24. The two prime numbers are 883 and 859. We can begin by dividing 1742

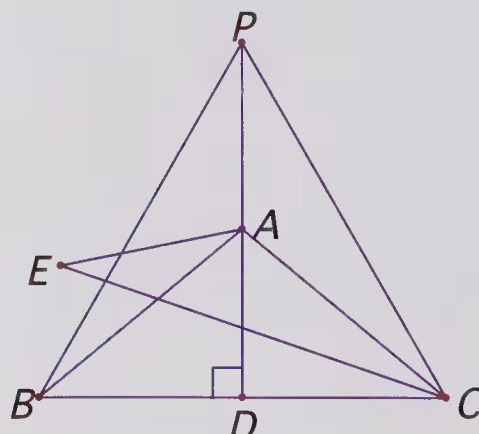
in half to get 871, which is divisible by 13. The primes surrounding 871 are 863 and 877. However, $1742 - 863 = 879$, which is not prime, and $1742 - 877 = 865$, which is also not prime. Thus, we must look at the next possible pair of primes—namely, 859 and 881. The partner of 881 is 861, which is not prime, but the partner of 859 is 883, which is prime and gives us the result $883 - 859 = 24$.

23. 36. Label the triangle so that the given side, AB , equals $20\sqrt{3}$ and the altitude is drawn from C to point D . Then $AC = 8$ since the hypotenuse in a 30° - 60° - 90° triangle is twice the side opposite the 30° angle. The length of the side opposite the 60° angle, AD , is $4\sqrt{3}$. Therefore, $DB = 20\sqrt{3} - 4\sqrt{3} = 16\sqrt{3}$. Apply the Pythagorean theorem in $\triangle CDB$ to find the hypotenuse: $CB^2 = 4^2 + (16\sqrt{3})^2 = 16 + 768 = 784 \rightarrow CB = 28$. The sum of the two sides is $8 + 28 = 36$.



Alternate solution: After obtaining AC , use the law of cosines in $\triangle ABC$ to find CB .

24. 30° . The goal will be to prove that $\triangle ACE \cong \triangle ACP$ by SAS. We know that $PC = CE$ and $\angle PCA = \angle ACE$ by construction. By the reflexive property, $AC = AC$, so the triangles are congruent. Notice that $\triangle PBC$ is equilateral since P is equidistant from B and C . The altitude of an equilateral triangle is also an angle bisector, so the measure of $\angle CPD = 30^\circ = m\angle AEC$.



25. $1/4, 1/13, 1/156$. Notice that

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)},$$

so

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{4 \cdot 3} = \frac{1}{4} + \frac{1}{12}.$$

Repeat the application of the theorem for the fraction $1/12$:

$$\frac{1}{12} = \frac{1}{13} + \frac{1}{12 \cdot 13} = \frac{1}{13} + \frac{1}{156}$$

Finally, $1/3 = 1/4 + 1/13 + 1/156$. This solution is not unique because $1/3 = 1/5 + 1/12 + 1/20$. Notice that the theorem stated allows us to write any unit fraction as the sum of any finite number of other unit fractions.

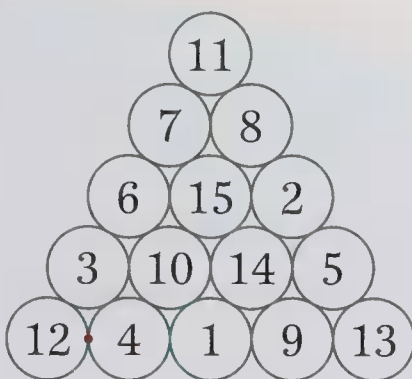
26. TODAY IS THURSDAY. No letter is assigned to 0, so the first number must be 20, which is the letter T. The next number could be 1 or 15, which is either A or O. If it is A, then the next number is 5, the letter E, followed by D. Since TAED is not a word, we know that the second number is 15 (O). The third number is 4, so the third letter is D. We now have the letters TOD. The fourth number can be 1 (A) or 12 (L). Reject the L and continue with the next number as 2 (B) or 25 (Y). Continue in this fashion to get to the result: TODAY IS THURSDAY.

27. Call the objects a, b, c , and d . Place a and b on one side of the scale and c and d on the other side. One side must be heavier—say, c and d —so one of these must be the 11 oz. object. Place c on one side of the scale and d on the other side. Whichever side is heavier holds the 11 oz. object.

28. Once you have determined the heaviest object—say, d —place a and b back on one side of the scale and d on the other. Three results are possible: (1) The objects balance, so a and b add to 11 and must be 3 oz. and 8 oz.; (2) the side with d is heavier, so a and b are 3 oz. and 5 oz.; or (3) the side with d is lighter, so a and b are 5 oz. and 8 oz.

The next weighing will determine the specific weight of a and b . Then there will be only one weight remaining for c .

29. This problem has many solutions. The first key is to realize that the three middle numbers are $\{10, 14, 15\}$, $\{11, 13, 15\}$, and $\{12, 13, 14\}$. The second key is to realize that the sum of the integers from 1 to 15, inclusive, is $(15)(16)/2 = 120$ but that the sum of four 39s is 156. Therefore, the difference $156 - 120 = 36$ must be the sum of the three vertices, which are each counted twice. Thus, the only possible vertices are $\{15, 14, 7\}$, $\{15, 13, 8\}$, $\{15, 12, 9\}$, $\{15, 11, 10\}$, $\{14, 13, 9\}$, $\{14, 12, 10\}$, and $\{13, 12, 11\}$. When we select one of the first two options for the center circles, we have only one corresponding option for the vertices. The third option listed for the center circles will provide three possible sets for the vertices. Once the three center circles and three vertices are filled, the nine remaining numbers become relatively easy to place.



30. $15/4$. The area of a triangle is the product of a side times the altitude to that side, so we can say that in any given triangle a side and its corresponding altitude are inversely proportional. If we name the shortest two sides a and b and the longest side c , we know by the triangle inequality theorem that $a + b > c$. Thus, $1/6 + 1/10 > 1/h \rightarrow 1/h < 8/30 = 4/15 \rightarrow h > 15/4$.



For an Excel spreadsheet for the solution for problem 1 for the September 2013 Calendar, go to www.nctm.org/mt.

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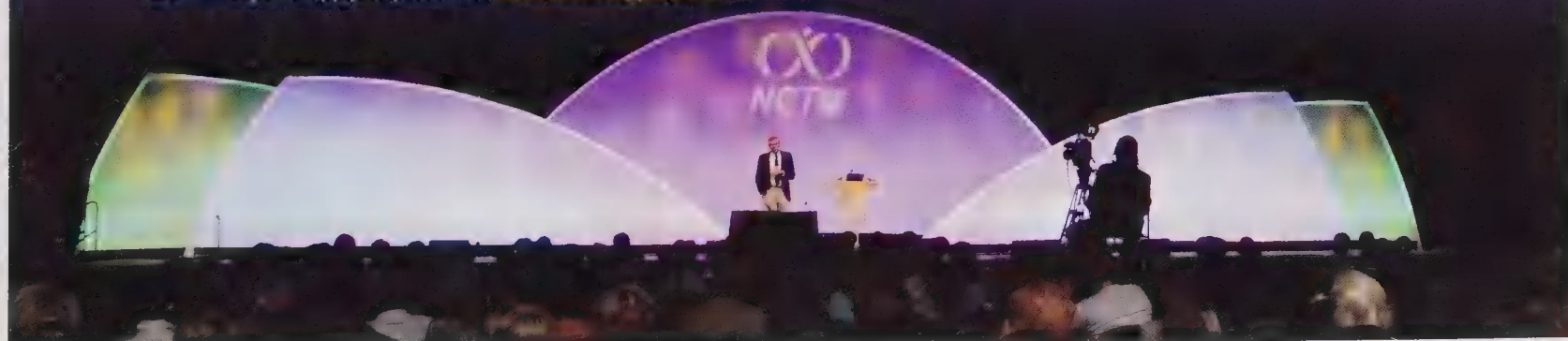
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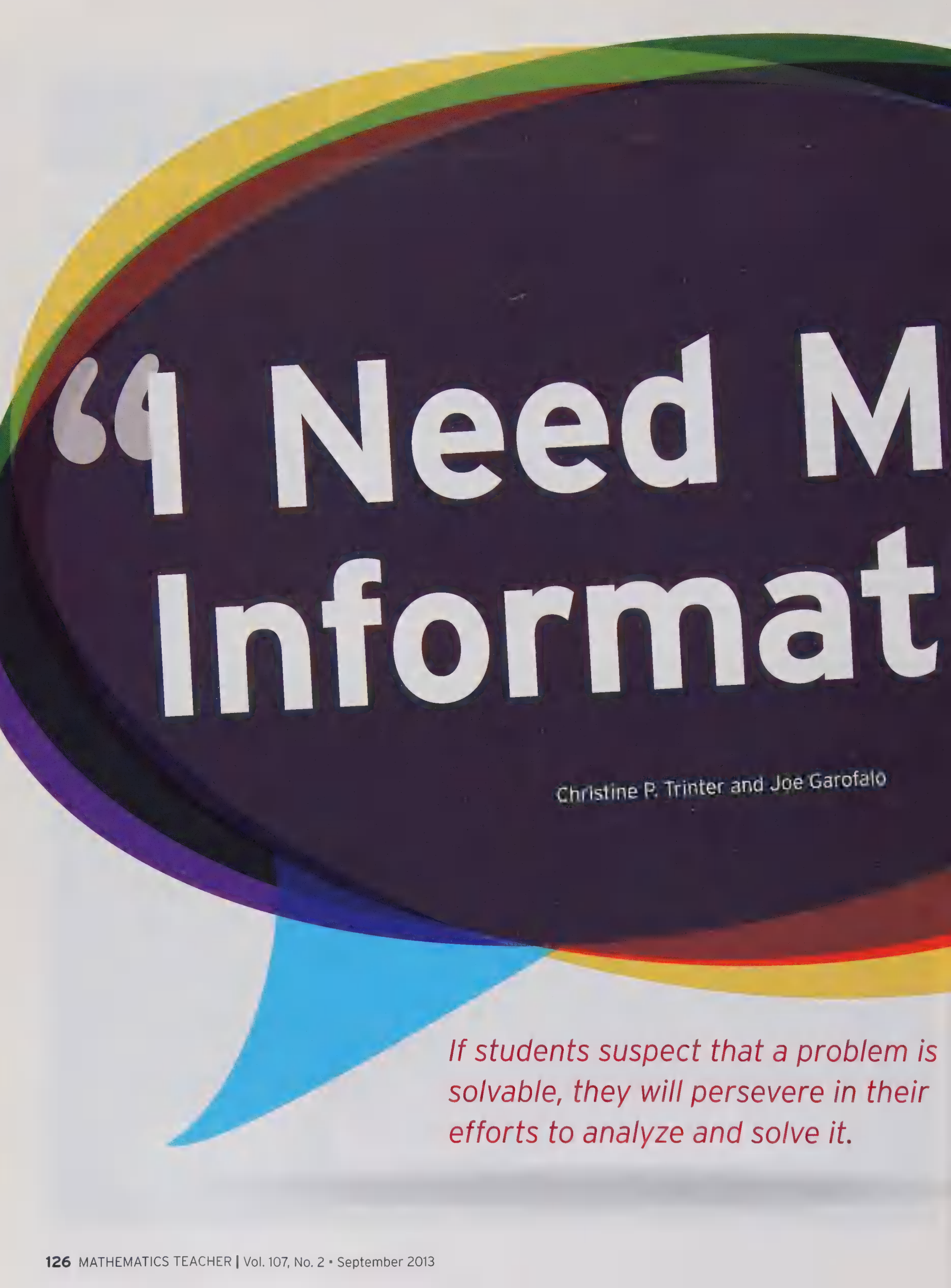
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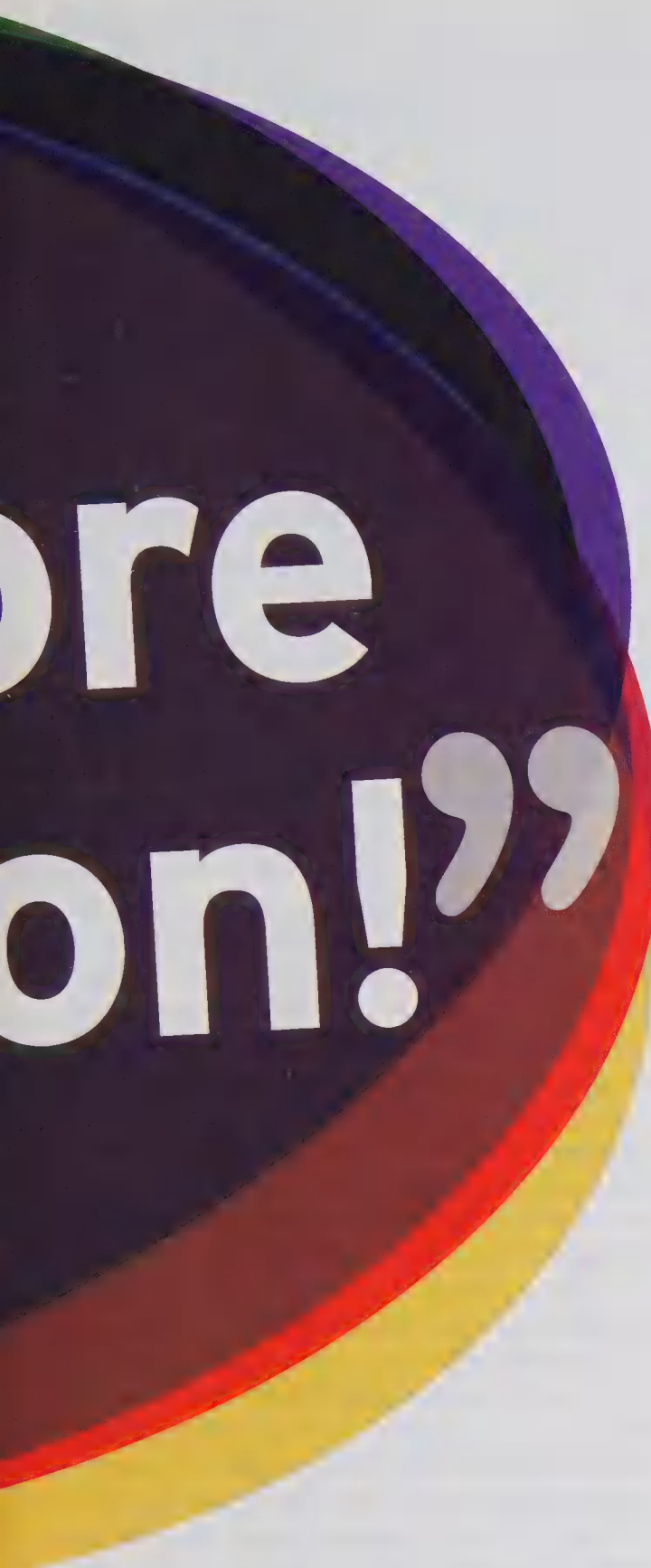




“I Need M Informat

Christine P. Trinter and Joe Garofalo

If students suspect that a problem is solvable, they will persevere in their efforts to analyze and solve it.



descriptions of some common reactions and solution strategies. These tasks address NCTM's Standards (2000) and the Common Core Standards for Mathematical Practice (CCSSI 2010). We encourage readers to solve the problems themselves to better appreciate the solutions and hints given.

TASK 1: THE PSYCHOLOGIST AND WIFE HANDSHAKING PROBLEM

There are 5 couples at a party. One of the people at the party is a psychologist who asks all the other guests how many people they shook hands with that evening. He receives a different answer from each person, ranging from 0 through 8. If we assume that husbands and wives do not shake hands with each other and that people do not shake hands with themselves, how many people did the psychologist's wife shake hands with? (Original problem in Hayes [1981], p. 65; modified by Trinter and Garofalo)

This task has elicited a variety of interesting responses and solution strategies from solvers. At first glance, many feel that not enough information has been provided to answer the question, but almost all persevere to find an answer. Their initial responses often place them into one of three groups: (1) people who try to distract from solving the problem; (2) frustrated solvers; and (3) solvers who enjoy the challenge.

The "distractors" are those who ask questions or make statements such as "Why would the psychologist ask everyone how many people they shook hands with?" or "His wife didn't shake hands with anyone because she was a germophobe." Some who respond this way may not know where to begin to solve the task; hence, they try to distract others from engaging with the problem or just give a superficial answer. A few become frustrated and give up.

Despite the few people who are intimidated by this task, most fall into the third category—solvers who enjoyed the challenge. These solvers become instantly engaged in the task, as evidenced in their lively conversations and animated interactions while working toward a solution. Four predominant solution strategies that we have seen from this group are acting out the scenario, creating a collection of pairs, drawing a visual representation, and constructing a matrix.

In a recent workshop, ten teachers stood in the middle of the room and attempted to model the task as if it were a play. A group leader directed each person to shake hands with others; she assigned herself the role of the psychologist. Because the "actors" did not devise a strategic plan (or write a screenplay), they were unsuccessful in finding a solution after a large number of handshakes.

When confronted with a challenging problem, many solvers may at first think that not enough information has been provided. If, however, they suspect that the problem is solvable, this feeling typically influences them to persevere as well as monitor and reflect on their efforts.

We present four nonroutine tasks that we have posed to mathematics teachers and students to engage them in problem analysis and strategic behavior. Included with two of the tasks are

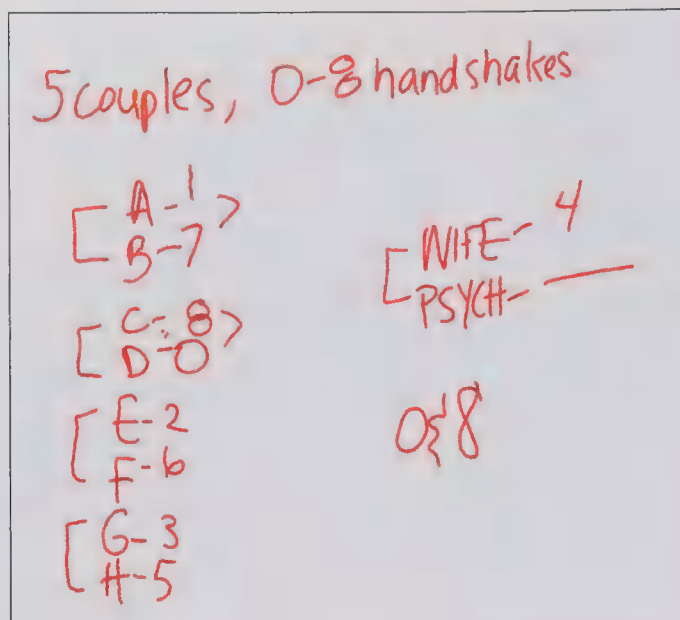


Fig. 1 Listing pairs of spouses can be a successful strategy for the Psychologist and Wife Handshaking problem.

Some solvers generate lists of pairs for the couples, sometimes without considering all the problem conditions. For these solvers, finding the solution can take many attempts. With each attempt, they realize the importance of each condition. For example, several teachers incorrectly paired the guest who replied “8 handshakes” with the guest who replied “1 handshake.” These teachers quickly assumed that the sum of each pair must be 9 because 9 people responded to the psychologist’s question. As a result, they learned a critical aspect of problem solving—consider carefully all the given information when devising a plan.

Solvers who take all the task information into account and reason from that information come to the solution more efficiently. For example, the student whose work is shown in **figure 1** realized that the person who shook hands with 1 other guest is married to the person who shook hands with 7 guests. Her reasoning follows: Both these guests shook the hand of the guest who shook hands 8 times; they did not shake one another’s hands; neither shook hands with the guest who shook hands 0 times; and the guest who shook hands 7 times must have shaken hands with everyone else, giving all the others at least 2 handshakes. This student’s work reflects similar reasoning for each pair and shows that the psychologist’s wife shook hands with 4 people. This method is easier for students if they focus first on the person who shook hands 8 times or 0 times.

Other solvers draw a visual representation of the problem in which they situate partygoers in a circular or even a linear format and draw lines to denote handshakes. Some using this approach forget to include the psychologist in their illustration, leading to incorrect solutions and confusion. With an accurate rendering, these solvers coupled the guests

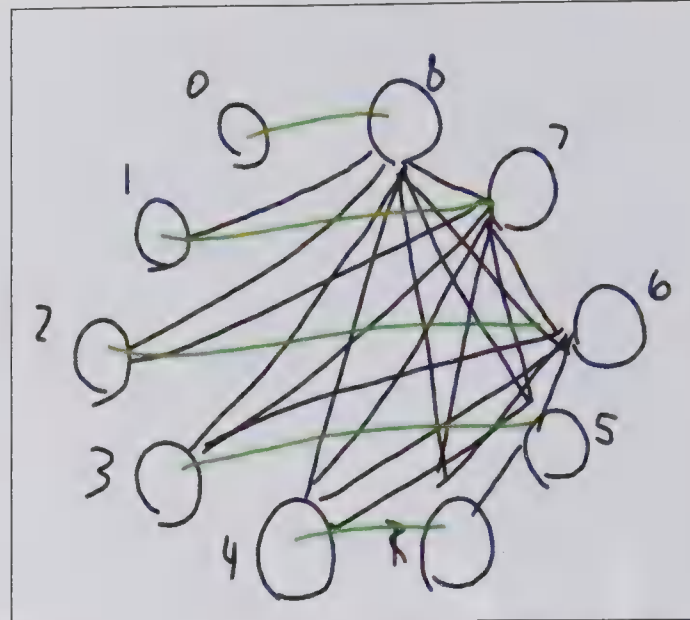


Fig. 2 A rendering of partygoers in a circular format helps solvers visualize the problem.

by using similar reasoning as in the listing-of-pairs strategy. The work of a student using this approach is shown in **figure 2**.

Hayes (1981) documented a fourth strategy (see **fig. 3**), which we have seen students use regularly. Using a matrix, Hayes arbitrarily assigned H_1 (husband 1) as the person who shook hands 8 times and drew Xs in each row and column to indicate the guests with whom H_1 shook hands. This approach revealed that H_1 ’s wife must not have shaken any hands with anyone because all the other wives shook hands with H_1 (as represented by the Xs); hence, none of the other wives could have answered zero to the psychologist’s initial question. Hayes then assigned H_2 as the person who shook hands 7 times. Completing the matrix with Xs and Os in this fashion uncovered the pattern that led to the same solution as the strategies shown in **figures 1** and **2**.

This task challenges solvers to think about well-known tasks, such as handshake exercises, in different ways. Finding the correct solution requires solvers to consider all the information provided, reason through the situation, persist, and reflect on their work—all valuable lessons in becoming an effective problem solver. And solvers have the freedom to use different representations and solution strategies. In our experience, when solvers discussed and compared their strategies, they were able to make links between the strategies and show the pattern in different ways. Further, they understand that enough information was provided.

TASK 2: THE CROSSING THE TRAIN BRIDGE PROBLEM

A man walks $\frac{2}{3}$ of the distance across a railroad bridge when he sees a train approaching him head-on at the rate of 45 mph. If he can just

Matrix Solution (Partial)										
	H ₁	W ₁	H ₂	W ₂	H ₃	W ₃	H ₄	W ₄	H ₅	W ₅
H ₁	O	O	X	X						
W ₁	O	O	O	O						
H ₂	X	O	O	O						
W ₂	X	O	O							
H ₃	X	O	X							
W ₃	X	O	X							
H ₄	X	O	X							
W ₄	X	O	X							
H ₅	X	O	X							
W ₅	X	O	X							

Source: Hayes (1981), p. 67

Fig. 3 Hayes's matrix also leads to a visual solution.

manage to escape being hit by the train by running to either end of the bridge, at what rate of speed does he run? (Origin unknown)

When faced with an unusual problem situation such as this one, most people fall back on prior knowledge and familiar strategies. Hence, the majority of solvers begin this task as they would a standard distance = rate • time task. In doing so, because they want to know the distance from the train to the bridge, some conclude that they do not have enough information to determine the man's speed. Some begin with a diagram and then do not know how to proceed (see fig. 4).

After carefully analyzing the problem situation, many solvers successfully use an algebraic method connected to a diagram to obtain a solution. Figure 5 illustrates a preservice teacher's illustration of the problem situation that includes the distance from the train to the bridge, the speed of the train, the man's two running options, and notation depicting the man's running speed. This sketch is an efficient model for representing these components. This solver did not get bogged down in the details of the illustration, a trap that some fall into.

The preservice teacher who solved this task set up algebraic proportions for each of the man's running options and used one of these proportions to create an expression for the distance from the train to the bridge. Specifically, the right column in figure 5 shows that time of the runner (his distance divided by his speed) set equal to the time of the train, where R_s represents the runner's speed, b the length of the bridge, and d the distance from the train to the bridge:

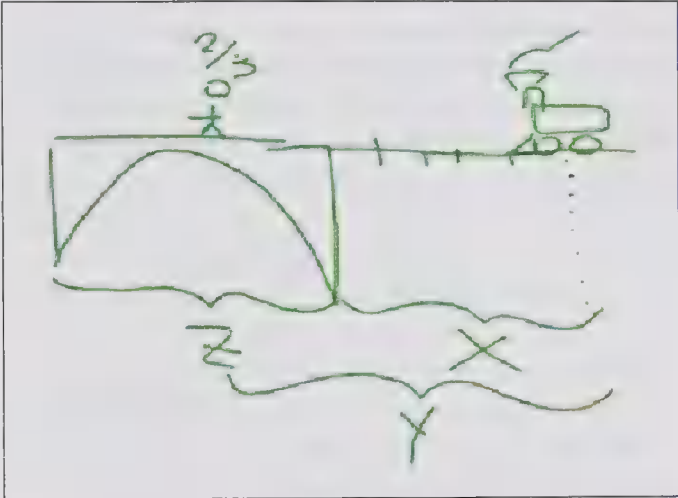


Fig. 4 A diagram such as this does not lead to a solution to the Crossing the Train Bridge problem.

$$\frac{\frac{2}{3}b}{R_s} = \frac{d+b}{45}$$

In solving this proportion for the distance from the train to the bridge, she represented the distance with the expression

$$d = \frac{30b}{R_s} - b.$$

She then substituted this expression into the proportion in the left column in figure 5 to solve for R_s . In particular, she set up a proportion representing the man's shorter option for running across $\frac{1}{3}$ of the bridge:

$$\left(\frac{\frac{1}{3}b}{R_s} = \frac{d}{45} \right)$$

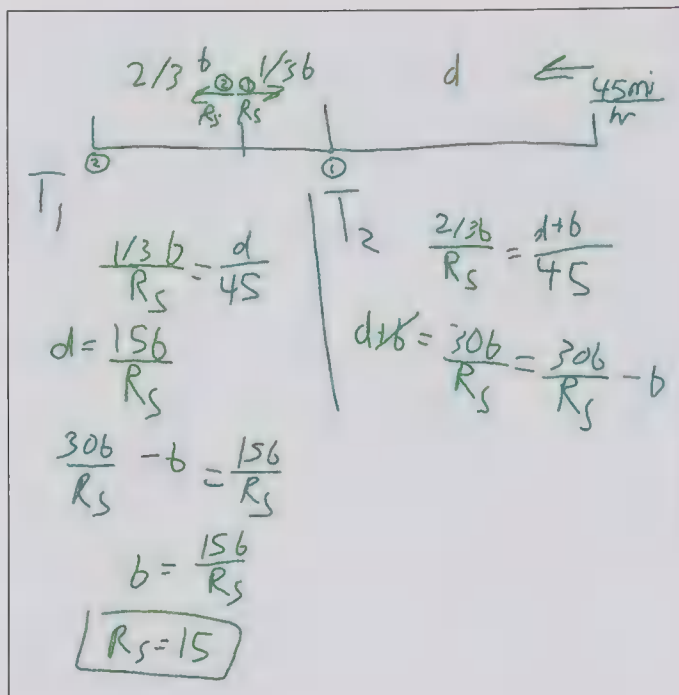


Fig. 5 This algebraic approach to solving the problem makes use of a diagram as well.

After isolating the variable representing distance between the train bridge and the train in this new proportion, she equated the two expressions that she found for d and manipulated the resulting equation—

$$\left(\frac{30b}{R_s} - b = \frac{15b}{R_s} \right)$$

—to find the result: $R_s = 15$. This solution was a lengthy algebraic method for finding the man’s speed.

Another preservice teacher used a more elegant strategy by focusing his attention on the diagram and reasoning from the information (see **fig. 6**). The solver drew a simple diagram that includes all the given information about the man’s distances to both ends of the bridge. He then indicated with arrows that when the train reaches point A, the man must also reach point A or point B.

This solver’s proportion

$$\frac{45}{1} = \frac{x}{\frac{1}{3}}$$

indicates that this situation can work only if the distance between the train and the bridge is equivalent to the length of the bridge itself. This is true because the time it takes for the man to run $1/3$ of the bridge is equal to the time it takes for the train to get to the edge of the bridge. If the man runs $1/3$ of the length of the bridge in the other direction (away from the train), the train just reaches the edge of the bridge and has the full bridge to travel while the man runs the other $1/3$ of the bridge; they must reach the far side at almost the same time

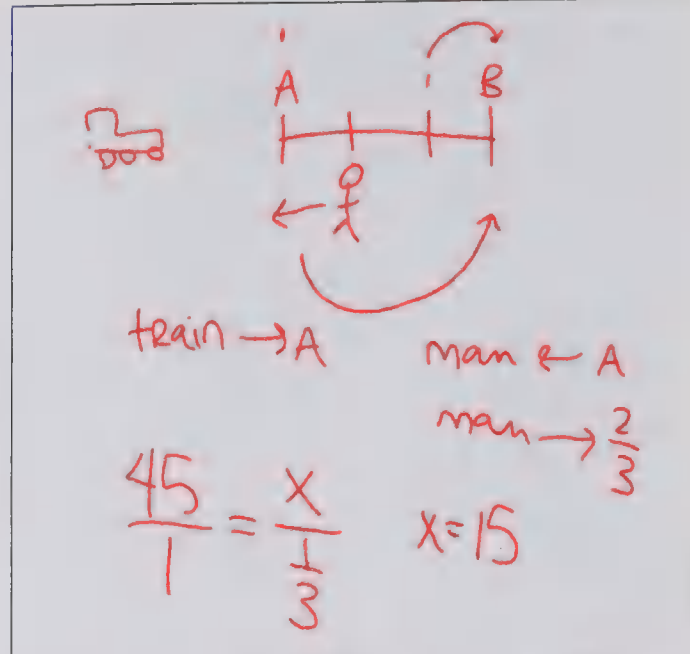


Fig. 6 Reasoning from the diagram and the information given can be a successful strategy.

(“he just manages to escape by running to either end”), meaning that the train must be a full bridge distance away. Therefore, the time it takes for the train to reach the end of the bridge is equal to the time it takes for the man to run $1/3$ of the bridge. Hence, the solver’s proportion shows the ratio of the train speed to the full bridge distance, which is equivalent to $1/3$ of the bridge distance to the man’s speed (x).

This more efficient strategy, which is not necessarily superior to the algebraic strategy described earlier, resulted from reasoning through the problem conditions with the aid of a simple representation. Often, successful algebraic solutions such as the one illustrated in **figure 5** do not result from a solver’s first algebraic attempt.

TASK 3: THE THREE DAUGHTERS PROBLEM

The host at a party said to his guests, “I have three daughters. The product of their ages is 72, and the sum of their ages is my house number.” He then asked: “How old is each daughter?” One guest rushed to the front door, looked at the house number, and told the host that he needed more information. The host replied, “The oldest one likes strawberry pudding.” The guest then announced their ages. How old is each daughter? (Origin unknown)

The key for many solvers is developing a list of possible house numbers based on a complete list of possible age combinations and noticing that one possible house number (14) can be obtained from two age combinations. However, some solvers first reason from the conditions. For example, recently a ninth-grade student immediately realized that “if he can’t tell, there must be two options to get his

house number” and found the solution after listing just five age combinations.

TASK 4: THE COMMUTER PROBLEM

Every day a commuter takes a train that arrives at her home station at 6:00 p.m. Every day her husband drives to the station to pick her up, always driving the same route at the same speed. One day the commuter gets off work early, takes an earlier train home, and arrives at her station at 5:00 p.m. Instead of waiting at the station for her husband, who left home at his usual time, she begins walking home. She meets her husband driving along his route, gets in the car, and they drive home. They arrive home 10 minutes earlier than usual. How long had the commuter been walking? (Origin unknown)

Many solvers draw a diagram and then try algebraic methods incorporating the distance = rate • time formula, often with several assumed distances and regular travel times. Some are successful with this approach, and some are not. However, as with task 2, it is simpler to reason from the conditions. In this case, arriving home 10 minutes earlier than usual means that the driver saved 5 minutes in each direction, and this realization leads to the determination that the commuter had walked for 55 minutes.

DISCUSSION

Use of these tasks supports both the NCTM and Common Core State Standards. NCTM’s (2000) Problem Solving Standard for grades 9–12 advocates:

In high school, students’ repertoires of problem-solving strategies expand significantly because students are capable of employing more-complex methods and their abilities to reflect on their knowledge and act accordingly have grown. Thus, students should emerge from high school with the disposition, knowledge, and strategies to deal with the new challenges they will encounter (p. 334).

Similarly, one of the Common Core Standards for Mathematical Practice (CCSSI 2010)—“Make sense of problems and persevere in solving them”—claims:

Mathematically proficient students . . . analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order

to gain insight into its solution. They monitor and evaluate their progress and change course if necessary (p. 6).

We have posed these four tasks presented here, and others like them, in classes, workshops, and presentations and found that solvers exhibit the following responses and outcomes after working on them:

1. Most students and teachers enjoy being challenged when they believe that they can be ultimately successful. Subsequently, most solvers persevere with these tasks.
2. Solvers learn that analyzing the meaning of problem conditions and reasoning from them are good habits.
3. Solvers become more fluent in creating representations; in addition, they become more flexible and comfortable working from them.
4. Solvers learn to assess their strategies and progress. They realize that quickly jumping to generate equations is not always the most efficient approach.
5. Solvers share their solution strategies, both successful and unsuccessful. Subsequently, they pick up strategies from one another to build their repertoires.
6. Solvers develop healthy dispositions, including “the confidence and willingness to take on new and difficult tasks” (NCTM 2000, p. 334).

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CHRISTINE P. TRINTER, ctrinter@vcu.edu, is an assistant professor of secondary mathematics education at Virginia Commonwealth University in Richmond. She is interested in problem solving and using technology and multiple representations to support students’ mathematical understanding. **JOE GAROFALO**,



jg2e@virginia.edu, is codirector of the Center for Technology and Teacher Education at the University of Virginia in Charlottesville. His interests include mathematical problem solving, educational technology, and mathematics teacher education.

Exploring Function Transformations Using the Common Core

Tying your teaching approach to the Common Core Standard for Geometry and Congruence will help students understand why functions behave as they do.

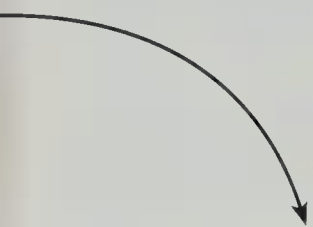
Becky Hall and Rich Giacin

When examining transformations of the plane in geometry, teachers typically have students experiment with transformations of polygons. Students are usually quick to notice patterns with ordered pairs.

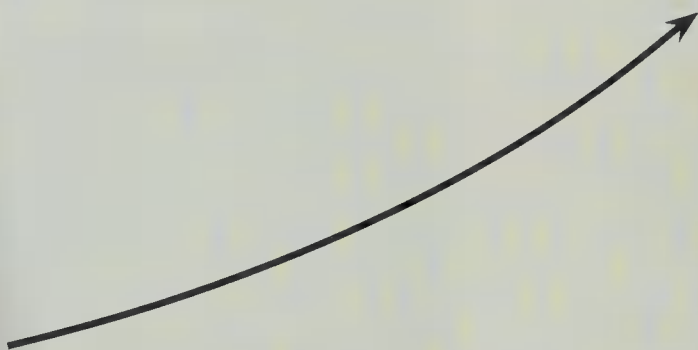
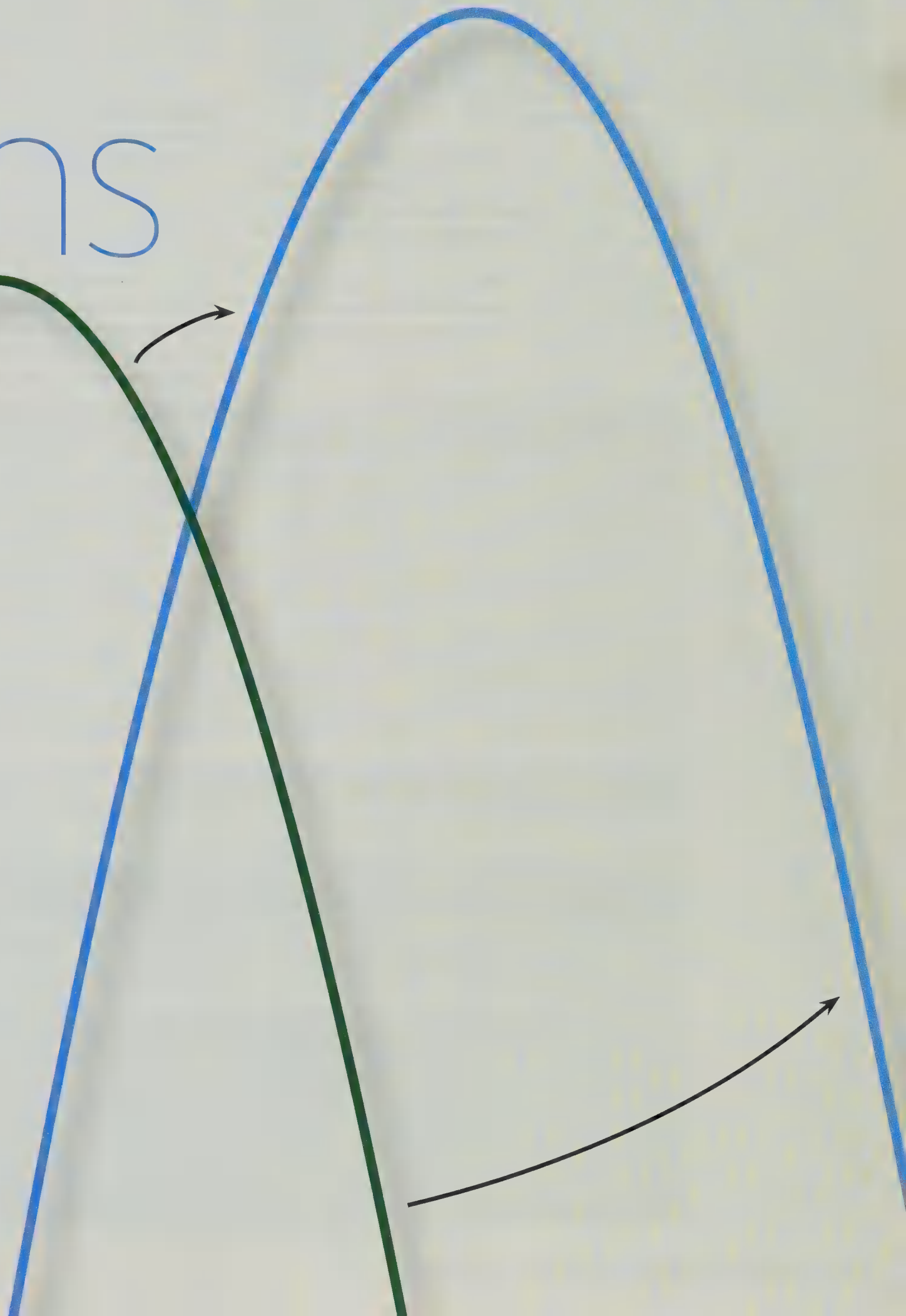
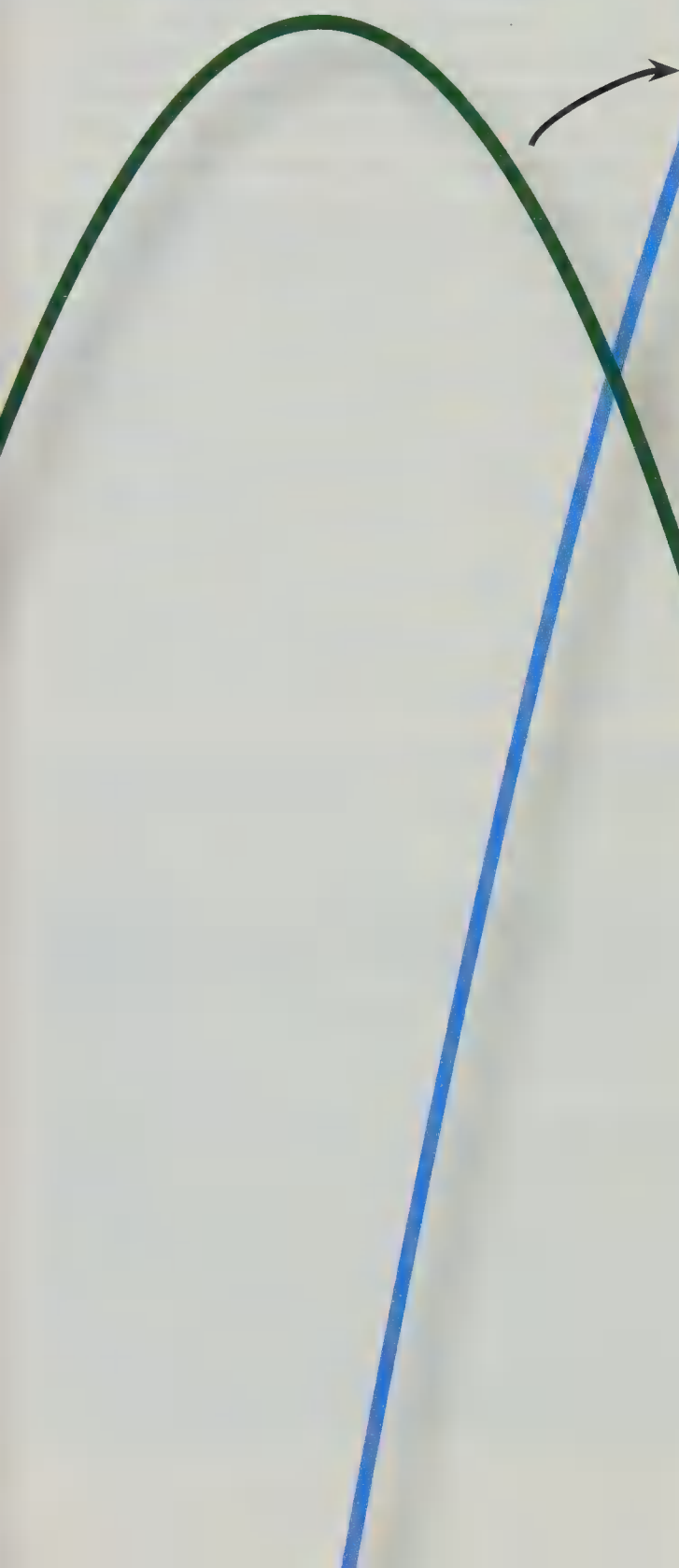
For example, ask students to translate a polygon three units to the right and compare the ordered pairs of the vertices of the original polygon with those of its image. They will discover that the x -coordinates have increased by three units. The Common Core State Standard, Geometry, Congruence 2 (G-CO.2), requires students to “describe transformations as

functions that take points in the plane as inputs and give other points as outputs” (p. 76). Viewing this translation as a function, as this standard requires, we obtain the function $T(x, y) = (x + 3, y)$. Students easily understand this function because adding to the x -coordinate would logically move the object right.

However, once students begin transforming functions, they often lose this understanding and resort to memorizing a list of rules that describe transformations. For example, students are told (or perhaps discover a pattern by graphing several examples) that translating $y = f(x)$ right three units results in the function $y = f(x - 3)$. Students may



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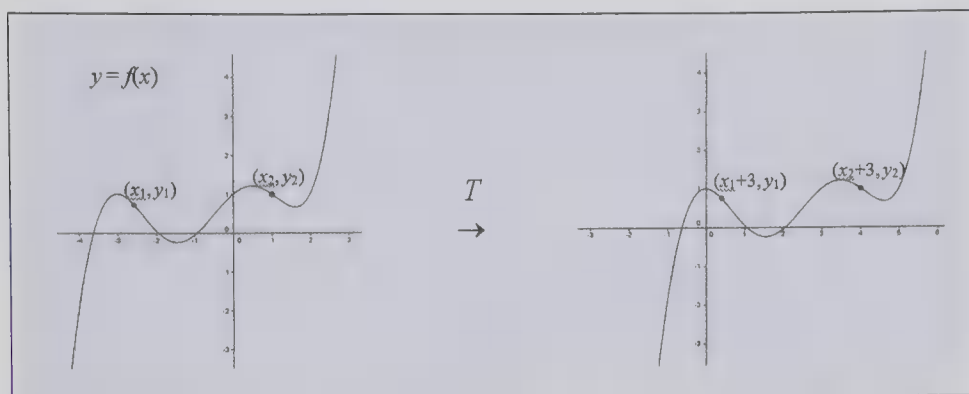


Fig. 1 The transformation function T maps (x, y) to $(x + 3, y)$.

ask why this statement is true; it seems counterintuitive. How many times have you answered this question with, “Whatever happens on the inside of the function does the opposite of what you think it will”? We admit that in the past we have resorted to a similar answer. The connection to ordered pairs explored in this Common Core standard can illuminate transformations on functions and help all teachers answer this question more effectively.

TRANSFORMATIONS AFFECTING THE x -COORDINATE

To make the explicit connection between transformations of ordered pairs and transformations of functions, we begin with the example highlighted in the opening paragraph. The function defined by $T(x, y) = (x + 3, y)$ takes points in the plane and shifts them 3 units to the right. But now let’s consider what effect this transformation has on the function $y = f(x)$.

Students may need to be reminded that they can view the function $y = f(x)$ as the set of all ordered pairs $(x, f(x))$. This step is necessary because the transformation function, T , inputs ordered pairs, not equations. The image of $T(x, f(x))$ is an ordered pair whose first coordinate is $(x + 3)$ and whose second coordinate is $f(x)$ —that is, $T(x, f(x)) = (x + 3, f(x))$ (see **fig. 1**).

You may want to remind students that we can

view the graph of the function $y = f(x)$ as the set of all ordered pairs $(x, f(x))$, because students may not be used to viewing a function whose input is $x + 3$. Instead, we want the input of the function to be a single variable. Given $T(x, f(x)) = (x + 3, f(x))$, we may use the simple substitution $x' = x + 3$, so $x' - 3 = x$, and rewrite the function as $T(x, f(x)) = (x', f(x' - 3))$. Thus, the transformation T takes $y = f(x)$ and maps it to $y = f(x' - 3)$. This process is outlined in **table 1**.

Students verify that this function will output $f(x)$ with an input of $x + 3$. This example illustrates why subtracting 3 “inside” the function translates it 3 units to the right.

Note that the algebra required to solve for x in the simple substitution above is key to illuminating why function transformations behave the way they do. Students already know that algebra helps “undo” operations to isolate a variable. This “undoing” is why some function transformations seem to behave in an “opposite” manner. This connection to algebra, we believe, will prevent students from viewing the method described here as just another procedure to memorize.

Consider a second example. The function defined by $S(x, y) = (2x, y)$ takes points in the plane and doubles their x -coordinate—that is, it horizontally stretches their location by a factor of 2. Again, students can easily understand this function; they can see that doubling the x -coordinate will stretch the original object horizontally. Under this transformation, $S(x, f(x)) = (2x, f(x))$ (see **fig. 2**).

To view the ordered pair $(2x, f(x))$ as an equation, we write it in a more conventional way with the substitution $x' = 2x$, so that $x = x'/2$. Thus, $S(x, f(x)) = (x', f(x'/2))$, and we see that the function $y = f(x)$ maps to $y = f(x'/2)$. Notice that this new function takes $2x$ as its input and outputs y . We outline this procedure in **table 2**.

The two examples discussed here are transformations that affect only the x -coordinates of an

Table 1 Summarizing the Process for a Horizontal Shift

Description	Ordered Pair Notation	Function Notation
A function that takes x as its input and outputs $f(x)$	$(x, f(x))$	$y = f(x)$
If we apply T (translating right 3 units) to the function above, we get the following:		
A function that takes $x + 3$ as its input and outputs $f(x)$	$(x + 3, f(x))$	Not possible without a substitution
To write the expression above into function notation, we apply the substitution $x' = x + 3$; thus, $x = x' - 3$.		
A function that takes x' as its input and outputs $f(x' - 3)$	$(x', f(x' - 3))$	$y = f(x' - 3)$

object. In both cases, the algebra required to solve for x' demonstrates the transformation's effect on the function. These transformations cause the most confusion for students because the functions seem to be the opposite of what they expect. More examples are listed in **table 3**.

TRANSFORMATIONS AFFECTING THE y -COORDINATE

This idea can be applied to transformations that affect the y -coordinate as well. For example, the function defined by $R(x, y) = (x, -y)$ switches the sign of the y -coordinate so the point is reflected over the x -axis. Under this transformation, $R(x, f(x)) = (x, -f(x))$. This is already written with a single variable input; thus, we see that the function $y = f(x)$ maps to $y = -f(x)$ because this is a function that takes x as its input and outputs $-f(x)$. No substitution was necessary in this problem (we could also let $x' = x$); for this reason, this transformation usually does not cause much confusion for students. More examples appear in **table 4**.

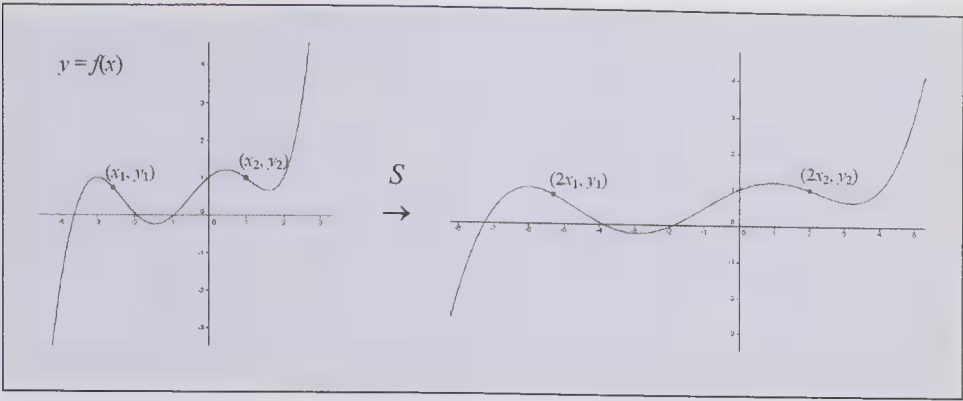


Fig. 2 The transformation function S maps (x, y) to $(2x, y)$.

COMPOSING FUNCTION TRANSFORMATIONS

This idea can be taken to a deeper level. As NCTM's Algebra Standard for grades 9–12 requires, we want students to be able to “understand and perform transformations such as . . . composing . . . commonly used functions . . .” (NCTM 2000, p. 296). If we use the method described here, performing multiple transformations becomes simply an exercise in composition

Table 2 Summarizing the Process for a Horizontal Stretch		
Description	Ordered Pair Notation	Function Notation
A function that takes x as its input and outputs $f(x)$	$(x, f(x))$	$y = f(x)$
If we apply S to the function above (i.e., if we stretch it horizontally by a factor of 2), we get the following:		
A function that takes $2x$ as its input and outputs $f(x)$	$(2x, f(x))$	Not possible without a substitution
To write the expression above into function notation, we apply the substitution $x' = 2x$; thus, $x = x'/2$.		
A function that takes x' as its input and outputs $f(x'/2)$	$(x', f(x'/2))$	$y = f(x'/2)$

Table 3 More Transformations Affecting x			
Description	(x, y) Maps to	Substitution of the Input Variable	$y = f(x)$ Maps to
Translate left 2 units	$(x - 2, y)$	$x' = x - 2$	$y = f(x' + 2)$
Horizontally compress by a factor of 3	$(x/3, y)$	$x' = x/3$	$y = f(3x')$
Reflect over y -axis	$(-x, y)$	$x' = -x$	$y = f(-x')$

Table 4 Transformations Affecting y			
Description	(x, y) Maps to	Substitution of the Input Variable	$y = f(x)$ Maps to
Translate up 5 units	$(x, y + 5)$	$x' = x$	$y = f(x') + 5$
Vertically stretch by a factor of 2	$(x, 2y)$	$x' = x$	$y = 2f(x')$
Vertically compress by a factor of 2	$(x, y/2)$	$x' = x$	$y = f(x')/2$

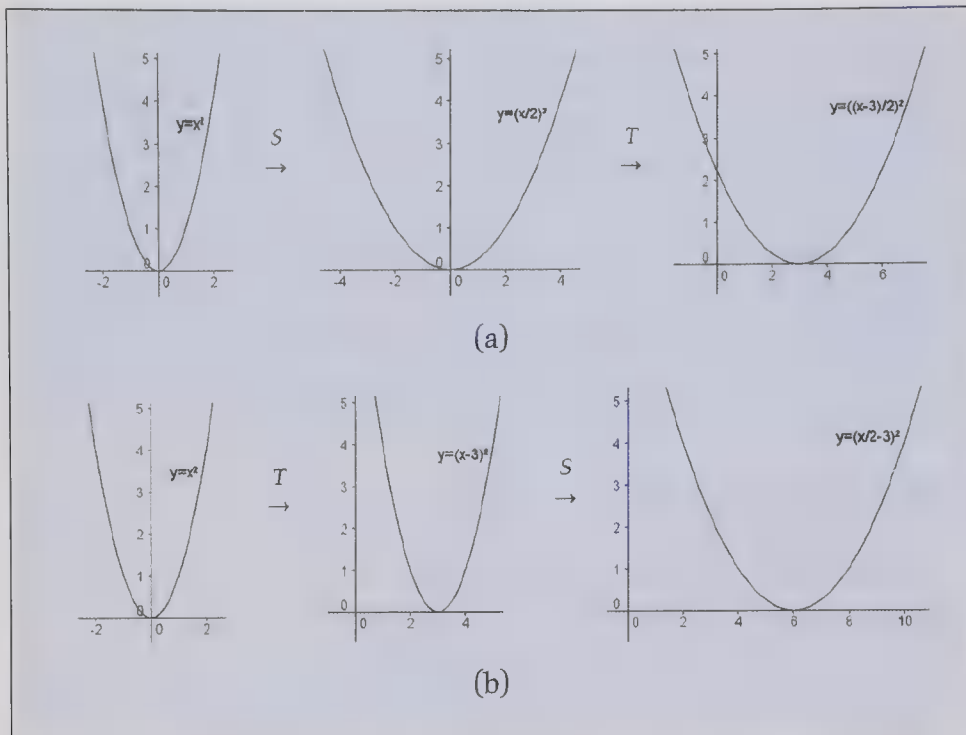


Fig. 3 Changing the order in which the transformation functions are composed sometimes produces different results.

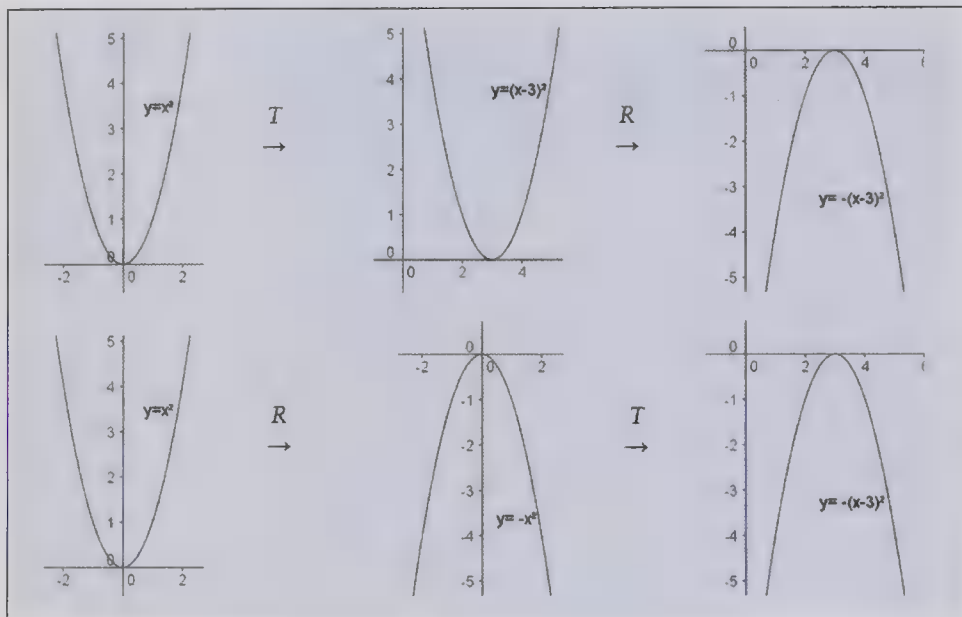


Fig. 4 Composing T and then R produces the same result as composing R and then T .

of functions. Recall that T is a transformation function that translates ordered pairs 3 units to the right, whereas S is a transformation function that horizontally stretches by a factor of 2. We can compose T with S . For example, $T(S(x, y)) = T(2x, y) = (2x + 3, y)$ takes points in the plane and horizontally stretches their location by a factor of 2 and then shifts them right 3 units.

Let's examine how this transformation affects the function $y = f(x)$, using the same process described earlier. Because $T(S(x, y)) = (2x + 3, f(x))$, we substitute $x' = 2x + 3$; thus, $x = (x' - 3)/2$, and $T(S(x, y)) = (x', f((x' - 3)/2))$. Therefore, the function $y = f(x)$ maps to $y = f((x' - 3)/2)$ because this is a function that takes $2x + 3$ as its input and outputs y . As another example, we can compose S and T in the reverse order to get $S(T(x, y)) =$

$S((x + 3), y) = (2(x + 3), y)$, which first shifts points in the plane right 3 units and then horizontally stretches their location by a factor of 2. So $S(T(x, f(x))) = (2(x + 3), f(x))$. Again, using the process described earlier, we let $x' = 2(x + 3)$, so that $x = x'/2 - 3$, and then let $S(T(x, f(x))) = (x', f(x'/2 - 3))$. Under this transformation, a function $y = f(x)$ maps to $y = f(x'/2 - 3)$.

Notice that the order in which we composed these transformation functions matters. If we first stretch and then translate, we get a result that differs from the result we would get if we first translate and then stretch. This concept may be clear to most readers, but it is not always clear to students. If it is not obvious, consider a particular example such as the function $f(x) = x^2$. Applying S to f gives us the new function $y = (x/2)^2$. Now applying T , we get the function $y = ((x - 3)/2)^2$ (see **fig. 3a**). However, first applying T to f gives $y = (x - 3)^2$. Then, applying S , we get the function $y = (x/2 - 3)^2$. Thus, applying S and then T (see **fig. 3a**) gives a result that is different from applying T and then S (see **fig. 3b**). Viewing these transformations graphically will make these ideas more concrete to students.

Now pose this question to students: Under what circumstances will the order in which we transform functions matter? In the example with S and T , both transformations change only the x -coordinate of the function, and we found that the order mattered. If we compose two transformation functions that change only the y -coordinate, the order will also matter. If, however, we compose two transformations in which one affects only the x -coordinate and the other affects only the y -coordinate, we will see that the order does not matter.

For example, first recall that R is a transformation function that reflects ordered pairs over the x -axis. Now, the order in which we compose R and T does not change the final result. Readers may want to perform these operations on the function $f(x) = x^2$ to verify this claim. They can then verify results by considering **figure 4**. Asking students to decide under what circumstances the order matters will help solidify their understanding of compositions of transformations.

STUDENT FEEDBACK

This material was presented to a group of ten high school students with various mathematical backgrounds. Included were freshmen, sophomores, juniors, and seniors then enrolled in courses from geometry to BC Calculus. The students were given written pre- and post-presentation questions to answer.

One question, contained on both questionnaires, was to compare the graph of $g(x) = (x - 3)^2$ with the graph of $f(x) = x^2$. For the preassessment, six

Preassessment Response	Postassessment Response
"I have no clue why. It just is."	"When you have $(x, f(x))$ and plug it into a value like A , the result will be $(x + 3, f(x))$ for $A((x, y)) = (x + 3, y)$. This equation shows that x is shifted 3 units to the right. However, when you translate back to $y = \underline{\hspace{1cm}}$ form, you will need to have $y = f(x - 3)$ to get $y = f(x)$ when $x + 3$ is plugged in for x ."
"These functions are based off of the parent function $y = a(x - h)^2 + k$. (h, k) is the vertex. So, for the $g(x)$ equation, the vertex is $(3, 0)$ and for the $f(x)$ equation, it is at $(0, 0)$. Both have a pattern of 1, 3, 5 ... because a indicates the pattern. They are upward facing because $a > 0$."	$ \begin{array}{c} "y = f(x) = x^2 \rightarrow (x, f(x)) \\ \downarrow T \\ g(x') = (x' - 3)^2 \leftarrow (x', f(x' - 3))" \end{array} $ Subtracting the 3 and putting the square outside the function 'undoes' the function, bringing it back to the original $f(x)$."

Fig. 5 A comparison of students' assessments shows greater understanding of function transformations.

of the ten students could describe the relationship between the two functions; however, when asked to explain why their answer was true, only two of the six could do so. Three other students stated the vertex form of a parabola and described the formula as a rule, and one student used a graphing calculator to verify her answer. For the postassessment, eight of the ten students could correctly describe the relationship and why it existed. Two students' responses to the question "Explain why shifting $f(x) = x^2$ 3 units to the right will give the graph of the function $g(x) = (x - 3)^2$ " are given in **figure 5**.

When we asked students what they learned, one student wrote, "Now I understand why you subtract 3 when moving to the right, other than just [because of] the formula." This is exactly the response we hoped for from students.

CONCLUSION

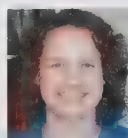
In most of the examples discussed here, transformations were performed on general functions, $y = f(x)$. Thus, the method we describe here can help students understand transformations on polynomials, trigonometric functions, logarithmic functions, exponential functions, or any other type of function. It can even be extended to nonfunctions, such as conic sections.

For both teachers and students, using the Common Core Standard for Geometry and Congruence will illuminate the behavior of function transformations and their compositions. When introducing transformations of functions or when asked why the function $y = f(x - 3)$ translates the function $y = f(x)$ three units to the right, teachers can make a connection to CCSS G-CO.2 and the notion of translating ordered pairs as demonstrated here. The

comparison of students' responses on pre- and post-assessments indicates that viewing transformations of functions using ordered pairs aids understanding. Students created a meaningful connection that illustrated why function transformations behave as they do.

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BECKY HALL, hallb@wcsu.edu, who previously taught high school mathematics, is an assistant professor at Western Connecticut State University in Danbury.



She teaches mathematics and secondary school education methods and serves as a university supervisor for secondary school student teachers. **RICH GIACINI**, giacini@newtown.k12.ct.us, a former naval officer and business executive, teaches mathematics at Newtown High School in Sandy Hook, Connecticut. He is a former MAT student of Hall's.

Not All Opportunities to Prove Are the Same

For many American students, high school geometry provides their only focused experience in writing proofs (Herbst 2002), and proof is often viewed as the application of recently learned theorems rather than a means of establishing and understanding the truth of general results (Soucy McCrone and Martin 2009).

Consider the following task:

Prove that at least one of the medians in any isosceles triangle is also an altitude.

Take a moment to think about how you might approach this task before continuing on. Your proof would likely start with a statement such as, “Let ABC be an isosceles triangle with sides AB and BC the congruent sides.” A series of properties would probably follow, leading to the conclusion that the median whose vertex is the intersection of the two congruent sides is also an altitude.

Now consider a similar task, with a sample proof included (see **fig. 1**). What differences do you notice between the two proving tasks? Are the prompts simply restatements of one another, or are they different in some meaningful way? Now consider the complete proofs. Are the two proofs the same or different? At first glance, the proofs may look nearly identical. In fact, a proof of the first claim may very well include all the statements in the worked example (see **fig. 1**), but it is also likely to contain an

extra line indicating the generic nature of isosceles triangle ABC —the “Let $ABC \dots$ ” step mentioned above.

Choosing an arbitrary representative to reason about is an important step in a proof dealing with a class of objects (such as isosceles triangles) because it allows one to reason about a particular object to justify claims about the entire set. In the worked example, the given triangle might be viewed as the particular triangle ABC in the diagram or as a representative of the class of all isosceles triangles. Further, the claims are different: The claim in the first task is general, whereas the claim in the worked example is particular.

Students may not notice a distinction between the claims associated with the two proving tasks. In addition, students may view the triangle given in **figure 1** as a particular triangle or as a representative of a general class of mathematical objects.

Imagine a typical high school geometry class in which students are discussing the median-altitude conjecture, with

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Edited by **Margaret Kinzel**, mkinzel@boise.state.edu

Boise State University, Boise, ID

Laurie Cavey, lauriecavey@boisestate.edu

Boise State University, Boise, ID

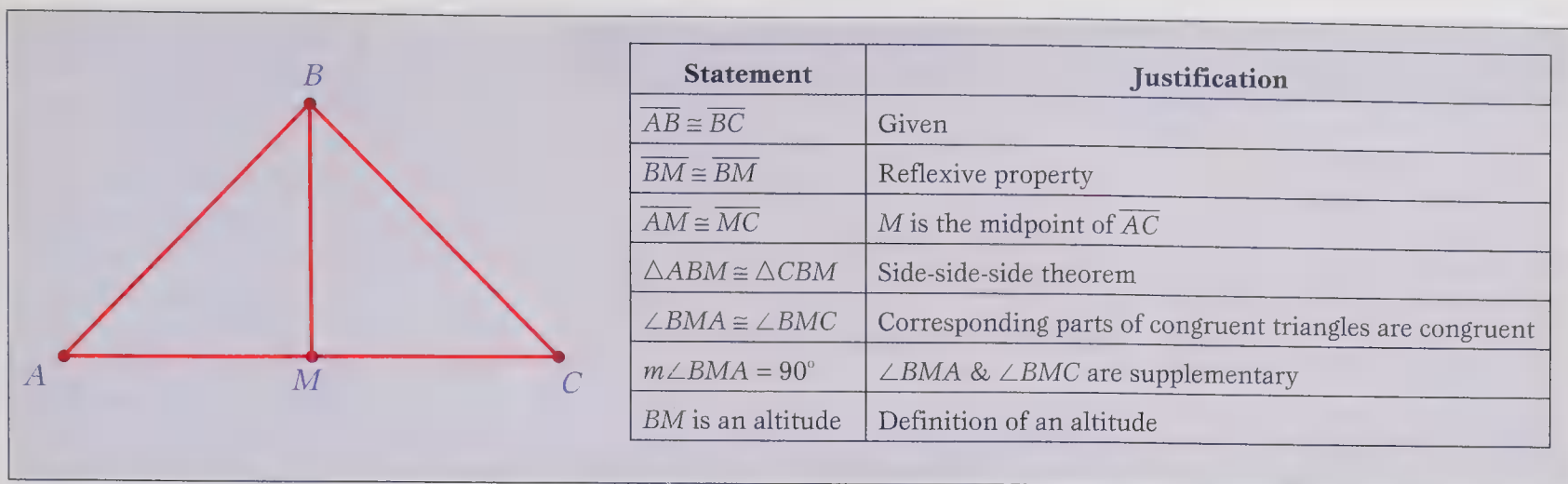


Fig. 1 A proof for the median-altitude conjecture does not indicate the generic nature of triangle ABC.

some students viewing the situation as particular and others viewing it as general. From the teacher's perspective, it may be quite difficult to identify how a student is thinking about these two very similar situations even if the student clearly describes his or her thinking. The matter may be more complicated for students because they may not even be aware of the perspective that differs from their own. The resulting discussion may appear quite coherent, but in reality students are talking past one another and thinking about the mathematical claim and argument in significantly different ways.

This imagined situation is not hypothetical. Researchers have documented that students do not always see the distinction between particular and general claims (Harel and Sowder 2007). For example, Chazan (1993) reported that some geometry students who proved a general claim believed that they had proven only the statement for the specific diagram used in the proof. Herbst and Brach (2006) found that some geometry students were able to distinguish between a general proof situation (such as the first situation that we posed about any isosceles triangle) and a particular proof situation (such as that shown in **fig. 1**). However, these students viewed only the particular problems as their responsibility as students; general claims and justifications were seen as being the responsibility of teachers or textbooks.

In addition, Buchbinder and Zaslavsky (2011) found that careful guidance toward a doubting mindset and purposeful selection of examples for claims were

needed to support students in negotiating the distinctions between particular and general proof situations. With these supports, the distinctions could lead to increased understanding of proof; without them, the distinctions can become a barrier to students' proof performance.

To move flexibly between the general and particular statements, students should be able to recognize at some broad level how these two statement types differ. Two major contributing factors that help shape students' opportunities to make this distinction are teachers and the textbooks they use. At the end of this article, we share suggestions about how to teach reasoning and proving, but first we look more deeply into the opportunities provided within textbooks for reasoning and proving. Although teacher implementation of textbook lessons varies widely (Tarr et al. 2008), textbooks play an important role in shaping students' opportunities to learn (Stein et al. 2007). In the next section, we provide findings from our study of reasoning-and-proving opportunities in high school geometry textbooks, which builds on other textbook analyses (Thompson et al. 2012; Davis 2010).

REASONING AND PROVING IN GEOMETRY TEXTBOOKS

To characterize reasoning-and-proving opportunities in high school geometry, we sampled six stand-alone—that is, nonintegrated—geometry textbooks. The textbooks we chose were *CME Geometry* (CME Project 2009); *Geometry* (Carter et al., Glencoe-McGraw Hill, 2010); *Geometry* (Burger et al., Holt McDougal, 2011); *Discovering*

Geometry (Serra, Key Curriculum, 2008); *Geometry* (Bass et al., Prentice Hall, 2009); and *Geometry* (Benson et al., UCSMP, 2009). These textbooks were chosen because they are from the most widely used series in the United States, accounting for approximately 90 percent of textbooks used in American classrooms (Dossey et al. 2008).

Within each textbook, we stratified the sample by chapter and included a minimum of 30 percent of the canonical sections from each chapter as well as the chapter review. In a chapter with seven sections, for example, we included four of the sections in our sample to meet the minimum 30 percent threshold. This stratification resulted in actual coding of 44 percent of sections across textbooks. Each canonical section was broken into roughly two parts—the exposition and the exercises. The exposition typically included worked examples and author explanations, whereas the student exercises are problems intended for students to work.

Working from the frameworks of Johnson and colleagues (2010) and Stylianides (2009), we coded all statements and exercises that addressed reasoning and proving in the sampled sections (for more details about the framework used, see Otten et al. forthcoming). Reasoning-and-proving activities were construed broadly to include the formulation of claims or conjectures, investigations or explanations of claims, and more formal proving activities. We also coded statements as to their location in the text, whether they occurred in the exposition (the beginning narrative part of a lesson) or the student exercises.

Table 1 Exercises and Proportion of Reasoning-and-Proving Exercises within Sampled Section

	Textbook					
	CME	Glencoe	Holt	Key	Prentice	UCSMP
Total number of exercises reviewed	1058	2730	2531	1489	2479	2181
Percentage related to reasoning and proving	27.6%	24.3%	23.6%	26.7%	19.5%	27.6%

	CME	Glencoe	Holt	Key	Prentice	UCSMP
Exposition	73%	69%	66%	83%	75%	72%
Exercises	45%	27%	20%	44%	27%	36%

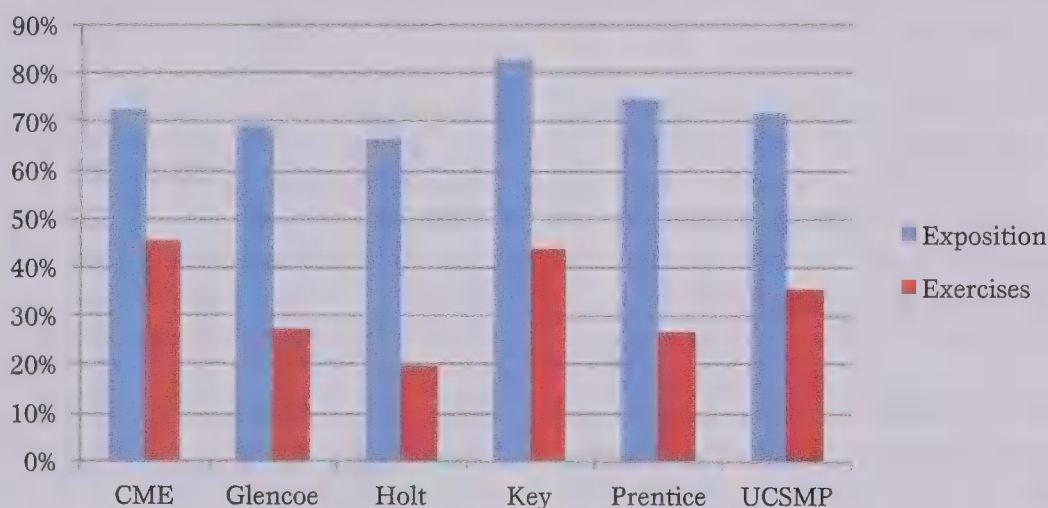


Fig. 2 The bar graph indicates the relative percentage of claims stated generally in the exposition and exercises in each textbook.

In addition, we coded the statement type (general or particular) and the type of activity (e.g., finding a counterexample, investigating a conjecture, writing a proof).

Because geometry courses have been the traditional home for deductive reasoning in the United States, we anticipated seeing many opportunities for students to reason and prove within the textbooks. **Table 1** shows the number of exercises in our sample within each textbook and the relative frequency of reasoning-and-proving exercises within each textbook. Although the raw number of exercises varied greatly among textbooks, the proportion of exercises that addressed reasoning and proving were generally around 20 to 25 percent across textbooks. These data indicate that thoughtful selection of homework exercises might result

in many opportunities for students to engage in reasoning and proving. They also indicate that less thoughtful selection of exercises could result in 75 to 80 percent of exercises having little to no opportunity for students to engage in reasoning and proving without additional teacher support.

The second result we share focuses on the types of claims that appear in the opportunities for reasoning and proving within the textbooks. The bar graph in **figure 2** shows the relative percentage of claims stated generally in the exposition and the relative percentage of claims stated generally in the student exercises.

One major trend is that in the exposition, students encountered explanations that were typically more general in nature. A major shift occurs in the exercises, however, where students are

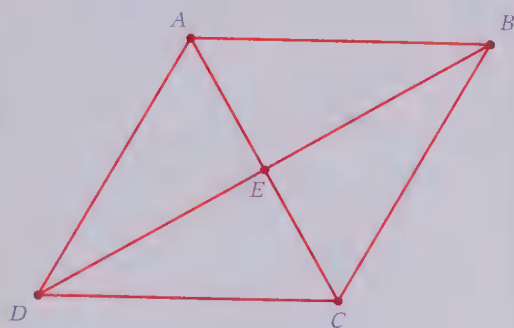
more likely to see problems that focus on particular situations rather than general ones. As a result, students are often told about general situations—for example, theorems—but then, in their homework, are asked to reason about situations that are particular in nature. This result may have important implications for how students view what it means to reason and prove in their geometry course. More specifically, because general situations necessitate the use of deductive reasoning, students may see the call to use deduction in problem situations that are not general simply as an arbitrary exercise instead of a means of establishing the validity of a statement.

These findings demonstrate how students might have relatively few opportunities to engage in reasoning and proving about general claims and thus few opportunities to see how general claims require different forms of reasoning (i.e., deductive reasoning) than do particular claims. Given the relatively low frequency of general claims in the exercises, it would not be surprising if students misinterpret general claims as particular ones. What, then, might a teacher do to support students in seeing the distinction and relationship between particular and general claims? In the next few sections, we provide some suggestions.

BRIDGING THE PARTICULAR-GENERAL DIVIDE

At this point, readers may be thinking that by critiquing the prevalence of particular situations in the student exercises, we advocate eliminating such opportunities from high school geometry textbooks or at least reducing them in favor of more general opportunities. Such an approach would likely be pedagogically disastrous. Particular reasoning-and-proving situations do play an important role in helping students connect mathematical ideas in a variety of situations. The problematic issue, we contend, is the minimal opportunities provided for students to reason about general cases compared with particular ones. In our analysis, however, we found cases that do provide opportunities for students to move from the particular to the general (such as that shown in **fig. 3**), what Mason and Pimm

Prove that a parallelogram with perpendicular diagonals is a rhombus.



Given: Parallelogram $ABCD$ with perpendicular diagonals AC and BD .

Prove: $ABCD$ is a rhombus.

Fig. 3 Some cases do exist where students can reason generally.

(1984) referred to as “seeing the general in the particular” (p. 277).

The problem in **figure 3** is representative of a statement type that we categorize as *bridging* because it invites discussion about how the general claim can be addressed with the specific figure given. Bridging statements appeared throughout all six textbooks that we analyzed, ranging from 2 percent (CME) to 10 percent (Glencoe). This statement type often occurred within the exercises (similar to those in **fig. 3**) or when a claim (e.g., a theorem) in the exposition was left to the student to prove later as an exercise.

The bridging statement in **figure 3** is general, but the authors have performed the important step of universal generalization—that is, selecting the arbitrary representative to reason about for the entire set of parallelograms. It is not clear how a student might interpret this type of problem and whether his or her proof would be about the particular parallelogram given or all parallelograms.

This lack of clarity could be seen as muddying the waters between statement types. Instead, this statement type (a general situation with a particular instantiation provided) could be used as an opportunity for teachers to talk with their students about the differences between particular and general claims. More specifically, bridging statements could be a starting point for discussing the distinctions between the particular figure and the general situation it rep-

resents as well as the different ways in which students could and likely do interpret the situation.

The data presented here provide an indication of the types of opportunities students have for reasoning and proving when using these textbooks. Although textbooks do play an important role in the types of learning opportunities available to students, the implementation of curricular materials is also critically important. Understanding the types of opportunities associated with different statement types may positively affect the way students use and view proof.

IMPLICATIONS FOR THE CLASSROOM

In the context of reasoning and proving, recognizing the distinction between general and particular situations is important because deductive reasoning allows one to make and justify claims about entire classes of mathematical objects. One cannot prove a claim about *all* isosceles triangles by simply doing a few examples and empirically reasoning about them—for example, through measuring. From a pedagogical perspective, general claims necessitate deductive reasoning because such claims cannot be proved in any other way. In contrast, if the situation is simply about some given triangle ABC , measuring seems like an appropriate and reasonable method, and asking for a proof might seem like an arbitrary request. Harel and Tall (1991) remind us that our goal as teachers should be to help students realize the intellectual necessity of deductive reasoning rather than just requiring them to engage in it.

In the curricula we analyzed and from our own experiences as teachers and students of high school geometry, we can safely make two assumptions: (1) Reasoning and proving is an integral part of the curriculum, and (2) the spatial nature of geometry necessitates diagrams to help with sense making. A single example can be powerful in helping students see the differences and connections between general and particular perspectives. Yet when examples are provided publicly, the level of particularity or generality is known best by its author, whether the textbook, the

teacher, or the student. Just because a teacher or textbook asks a student about a particular or general situation, we should not assume that the student will necessarily interpret the question as intended.

For this reason, when students or teachers present diagrams representing general relationships, they must pay attention to others’ possible particular interpretation of these diagrams. In other words, even if someone draws a specific figure and intends for it to represent an entire class, others may be thinking and reasoning about only that one object.

This is not to say that textbooks and teachers should never provide representations. Rather, we maintain that by empowering students to articulate the generality of a representation, students can be afforded more opportunities to justify the level of generality of their reasoning even if the given representation is particular in nature. One potentially rich resource that teachers should consider using is dynamic geometry software because it may motivate students to consider the need to represent a class of geometric objects (e.g., all isosceles triangles) with a single figure. By paying careful attention to student thinking and how students describe their thinking, teachers and students should be better positioned to uncover any inconsistencies between interpretations of a claim.

Because of the subtleties in both the distinctions and the interrelatedness of particular and general statements, research suggests the need for teachers to look at textbooks with an eye toward how some statement types may support or hinder the necessity for students to use deductive proof. Such analysis includes looking for places where students might misinterpret a general claim for a particular example and finding opportunities to extend the conversation from a particular problem to the more general case. By doing so, we anticipate that students will have a better chance to gain flexibility in reasoning across general and particular situations so that they might see the power in establishing mathematical truths through deductive reasoning.

ACKNOWLEDGMENTS

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NICHOLAS J. GILBERTSON, gilbe197@msu.edu, is a Ph.D. candidate in mathematics education at Michigan State University in East Lansing. He is interested in classroom discussions, curriculum, and teacher development.



SAMUEL OTTEN, ottensa@missouri.edu, is an assistant professor of mathematics education at the University of Missouri in Columbia. He is interested in secondary school students' participation in classroom discourse



and mathematical practices. **LORRAINE M. MALES**, lmales2@unl.edu, is an assistant professor of secondary school mathematics education at the University of Nebraska-Lincoln. She is interested in teacher development and the development, analysis, and use of curriculum materials. **D. LEE CLARK**, clarkd40@msu.edu, is a Ph.D. candidate in mathematics education at Michigan State University. He is interested in the making of and communication regarding policy in mathematics education.





Great Problems

“A great discovery solves a great problem, but there is
a grain of discovery in the solution of any problem”

— George Pólya (1957)

What makes a great problem?

The Editorial Panel of *Mathematics Teacher* wants you to share your great problems with other teachers. The Panel invites manuscripts that address the following questions:

- What is your favorite mathematical task? What makes it a great problem for students? What learning opportunities does this problem provide? What opportunities for student inquiry does this problem provide?
- What nonroutine mathematical tasks have you tried with students in grades 8–14 or in teacher education courses? How do such tasks connect to core school mathematics topics? How do these tasks open up opportunities for all students to make connections across mathematical ideas?
- Where do you find problems that drive great lessons? What specific multimedia resources or technologies could enrich teachers’ ability to pose mathematical questions in a classroom? How have you integrated such resources into your classroom instruction?

- What are examples of lessons or units that you have designed from great problems?

We would welcome regular feature articles or those submitted to departments such as Back Page: My Favorite Lesson; Activities for Students; Media Clips; Mathematical Lens; Delving Deeper; or Reader Reflections.

We are especially interested in mathematical tasks that engage students, promote sense making, and generate rich discussions and, in so doing, support NCTM’s Process Standards and the Common Core Standards for Mathematical Practice.

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Building Interactive Demonstrations with Sage

Sage is an open-source software package that can be used in many different areas of mathematics, ranging from algebra to calculus and beyond. One of the most exciting pedagogical features of Sage (<http://www.sagemath.org>) is its ability to create *interacts*—interactive examples that can be used in a classroom demonstration or by students in a computer laboratory. By accessing a simple Web-based Sage Notebook interface, we can quickly compute a diverse range of examples, such as finding the prime factorization of a positive integer or graphing transformations of functions. Graphing calculator explorations translate nicely into Sage, and the interact feature makes them much more dynamic.

Let's explore how to get started in Sage, go over a few basic commands, and then create some interactive demonstrations. We will touch on only a small fraction of Sage's powerful capabilities; additional resources are cited at the end of this article.

The simplest way to use Sage is through remote access of your own Sage Notebook account (freely available at <http://www.sagenb.org/>). Click on **Select an OpenID provider** and follow the prompts to set up a unique username and password (see **fig. 1**).

We will start by computing a simple multiplication problem, $3 \cdot 4$, in a new worksheet in the Sage Notebook. Once you are logged in, go to the menu in the upper-left corner and select **New worksheet**. A pop-up will appear in which you rename the worksheet, or you can just keep the name **Untitled** by hitting the return key.

The rectangle that appears in the new worksheet is an input cell, which is activated by clicking inside the cell. When activated, the input cell will appear outlined in blue. Enter $3 \cdot 4$ into the activated cell and then click on **evaluate** underneath the cell on the far left (see **fig. 2**).

Now let's look at some basic Sage commands—**factor**, **plot**, and **Combinations**. Enter the command **factor(1234)** into the next active cell and then click on **evaluate** (see **fig. 3**). The output, indented slightly below the command, gives the prime factorization of the integer 1234.

The **plot** command can be used to graph a function such as $f(x) = x^2$ on the

interval $[-1, 1]$. Type **plot(x^2,(x,-1,1))** into the cell highlighted in blue (see **fig. 4**). The second set of parentheses specifies the domain for the function. The syntax for plotting any function $f(x)$ on $[a, b]$ is **plot(f(x),(x,a,b))**.

Next we will illustrate the **Combinations** command along with two auxiliary commands: **show** and **list**. These commands allow us to find the combinations of any finite set of objects. As an illustration, we will find all the ways in which we can make sets from a list of three people—for example, Anna, Brian, and Casey. Begin by typing in the next cell **set=["Anna","Brian","Casey"]** and hit **return**. Do not evaluate the cell yet because we will enter another line that will use the set we just created. Type **show(Combinations(set).list())** immediately underneath the line **set=["Anna","Brian","Casey"]**, and then click on **evaluate** in the lower-left corner. The output (see **fig. 5**) shows all the groups that can be formed from the three people—Anna, Brian, and Casey—including the empty set, specified by the empty brackets: **[]**. The **show** command properly displays the output of the function, **Combinations**; the inclusion of **.list()** ensures that the output is made into a set itself.

Documentation for any Sage command can be found by typing the command name and then a question mark—such as **plot?**—into an active Sage Notebook cell and then clicking on **evaluate**. Tutorials with a full index of Sage commands are available online at

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more background information on the department and guidelines for submission, visit <http://www.nctm.org/publications/content.aspx?id=10440#tech>.

Edited by **Larry Ottman**
lottman@gfsnet.org
Germantown Friends School
Philadelphia, PA

James Kett
j.kett@gmail.com
Singapore-American High School (retired)
Singapore

<http://www.sagemath.org/doc/tutorial/>.

Now that we have a few Sage commands, we can feature them in an environment called an *interact*. Three diverse examples will showcase the powerful interactive aspects of Sage. The first interact demonstrates an exploration of number theoretic properties. Students can investigate questions such as these: Given a positive integer, n , is n prime or not? How do we answer that question efficiently as n gets large?

INTERACT 1: NUMBER THEORETIC PROPERTIES

To start an interactive display, type `@interact` into the input cell. Do not click on **evaluate**; rather, hit **return** and, directly underneath `@interact`, enter `def _(n=(1..1234)):`. Make sure that you insert a space immediately following `def` and then type an underscore before the open parenthesis. The underscore starts either the slider definition or the input box definition; a final colon separates the slider or input box from the function or functions used in the interact (see **fig. 6**). On the next line, type `print factor(n)` and then click on **evaluate**. If this line is not automatically indented as in **figure 6**, then indent it manually. These three lines allow us to find dynamically the prime factorization of any positive integer less than or equal to 1234. **Figure 6** shows the completed interact and the resulting factorization of 265.

Students can produce many interactive examples with Sage by clicking on and dragging the slider (highlighted in blue) or using the left and right arrow keys to alter the value of n . As the value of n changes, the output changes dynamically without students having to click on **evaluate** again.

In addition to possibilities for explorations through Sage in a prealgebra or algebra classroom, students could take the time at the beginning of a unit on polynomial factorization techniques to review integer factorization. They can build confidence and the numerical skills necessary to determine the greatest common factors from polynomials. For example, if we have the polynomial $107x^2 + 457x$, how do we know definitively that the greatest common factor of $107x^2$ and $457x$ is just x ? Students can examine such atypical

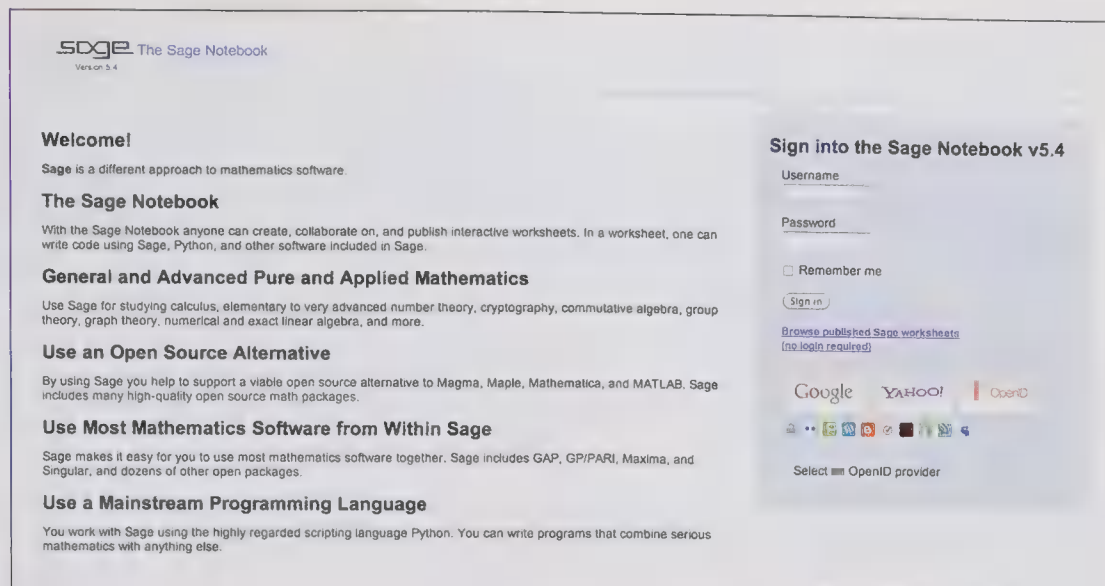


Fig. 1 At the online Sage Notebook sign-in screen, users must first choose an OpenID provider such as Google or Yahoo!

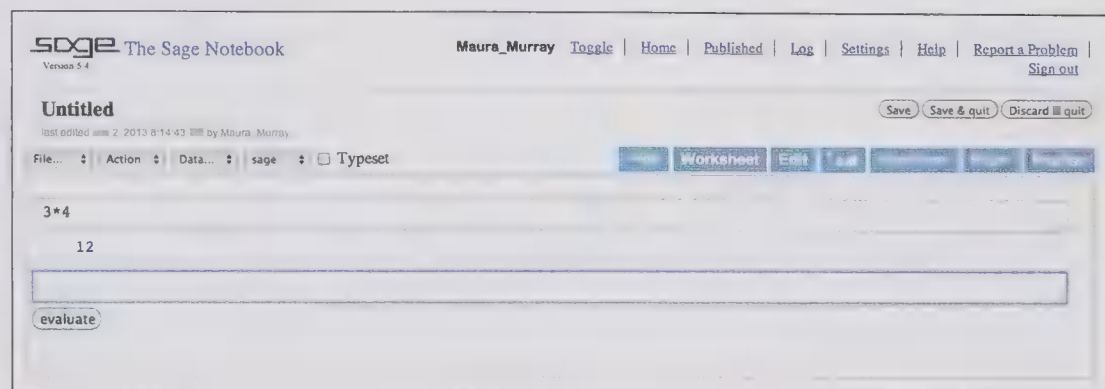


Fig. 2 The cell with the problem 3×4 is evaluated, and a new cell becomes available for input.

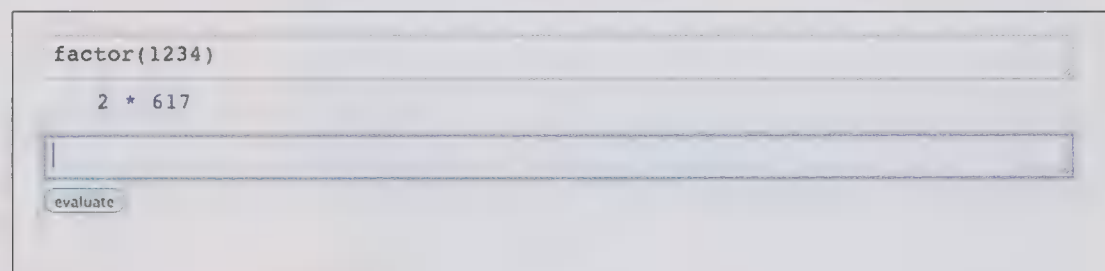


Fig. 3 The command `factor(1234)` produces the prime factorization of 1234.

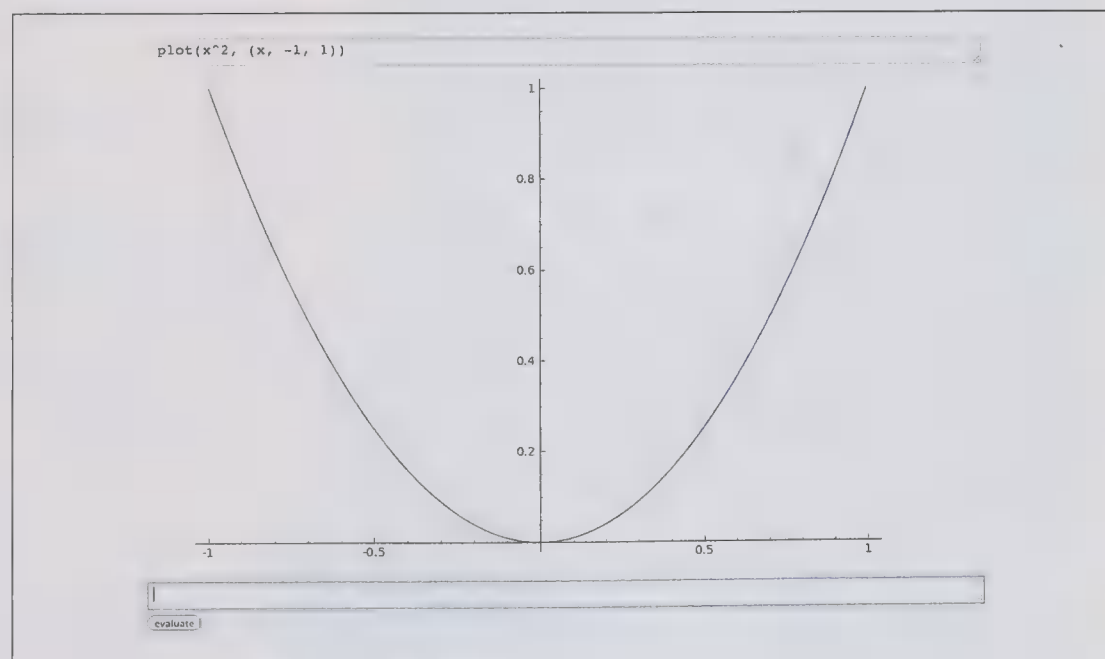


Fig. 4 The command `plot(x^2,(x,-1,1))` plots the function $f(x) = x^2$ on $[-1, 1]$.

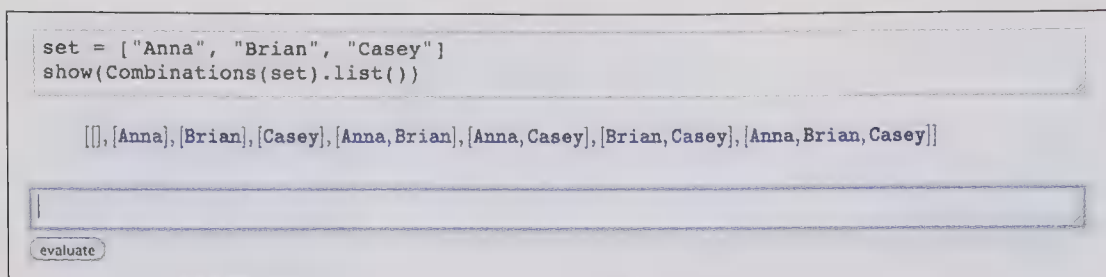


Fig. 5 A set of three people is defined, and then all possible groupings are found.

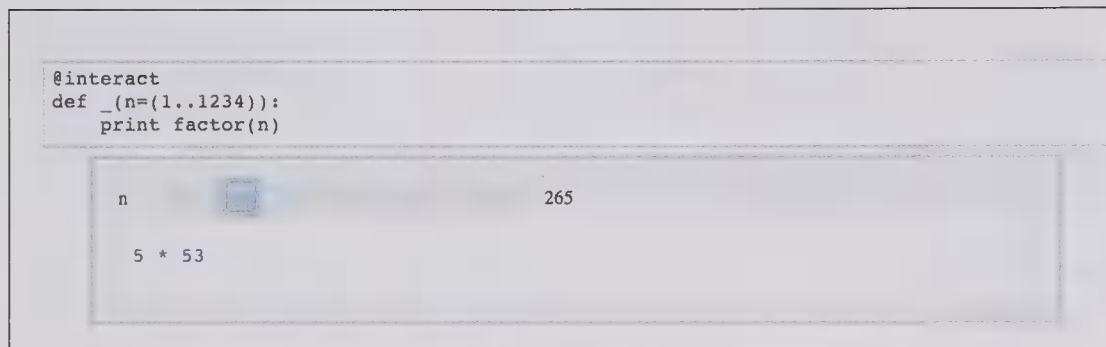


Fig. 6 This interact can be used to find the prime factorization of any integer between 1 and 1234.

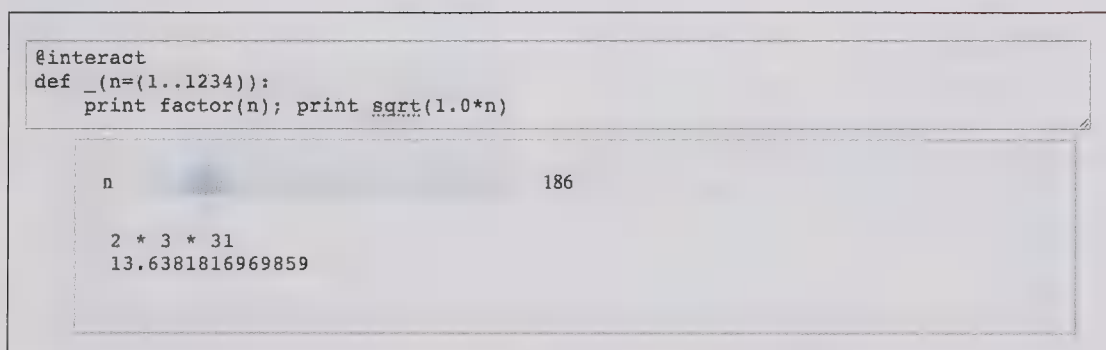


Fig. 7 The factoring interact has been modified to include $\sqrt{n}$ in the output.

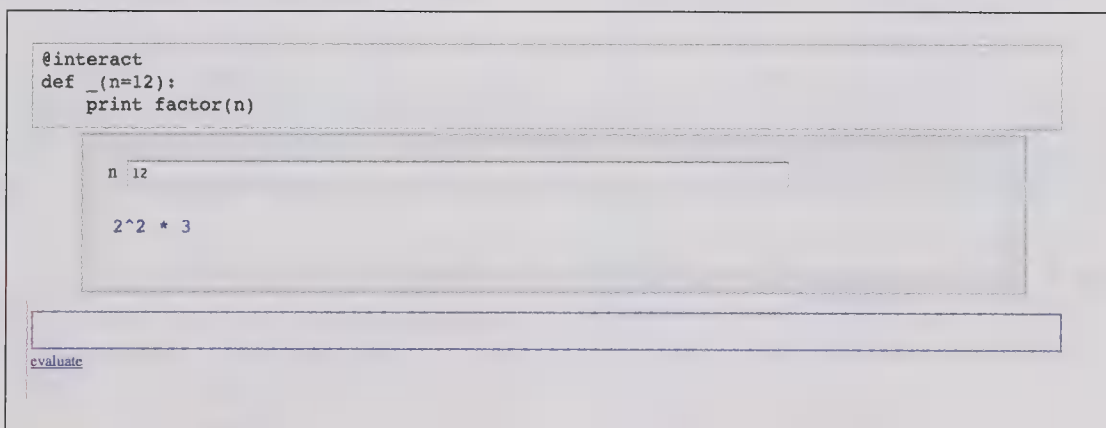


Fig. 8 The factoring interact now uses a dynamic input box rather than a slider.

coefficients with the use of Sage and ask questions about efficiently factoring any integer. Sage can quickly confirm that the integers 107 and 457 are prime, and students can verify these results manually by dividing by successive primes less than the number's square root.

The code can be modified to display the square root of n along with the prime factorization by clicking in the code cell until it is highlighted in blue and then inserting a semicolon after `factor(n)`, a space, and the command `print sqrt(1.0*n)`

(see **fig. 7**). Click on **evaluate** to make the changes active. The command `1.0*n` makes sure a decimal approximation of the square root is displayed. Otherwise, the command `sqrt(n)` returns the exact square root of n , which is not as useful for this exploration.

If you want to change the slider to an input box instead, start with `@interact` in a new cell, press **return**, and then type `def _(n=(12)):`. Press **return** and then `print factor(n)`. Clicking on **evaluate** creates an input box with the input 12, which you

can then change to other positive integers. To change the input, just click in the box, delete 12, insert another integer value, and then press **return** (see **fig. 8**).

INTERACT 2: EXPLORING GRAPHS AND FACTORIZATION

The second interact allows for dynamic exploration of the graph of $f(x) = ax^2 + bx + c$ and the factorization of $ax^2 + bx + c$. By looking at these two pieces of information simultaneously, students can better understand how values of x where $ax^2 + bx + c = 0$ relate to the graph of $f(x) = ax^2 + bx + c$. Students can investigate questions such as these: How do you know when $f(x)$ crosses the x -axis at integral values of x ? Can you predict when a quadratic will stay above or below the x -axis?

To start this interact, we will first need to define a , b , and c as variables outside the interact environment. First, enter `var('a')` in a new cell. Click on **return** without evaluating the cell and enter `var('b')` directly underneath `var('a')`. Press **return** and repeat the process for c . Hit **return** and type `@interact`. Then press **return** and enter `def_quadratic(a=1,b=2,c=-3):`.

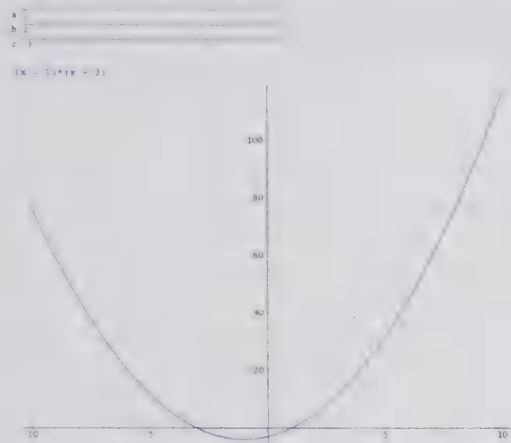
Now we are ready to define what the interact will do. Press **return** and type `print(factor(a*x^2+b*x+c))` on the next line. Note that multiplication is not implicit in this case, so you must include the `*`. Finally, press **return** and type `show(plot(a*x^2+b*x+c,(x,-10,10)))`. Before clicking on **evaluate** underneath the cell, check that all the appropriate parentheses are in place (see **fig. 9**).

The function $f(x) = ax^2 + bx + c$ is graphed in the window $(-10, 10)$ with initial values of $a = 1$, $b = 2$, and $c = -3$. Click in the input boxes next to a , b , and c and change the values there. Hit **return** or click in the next cell as you update each one, and the graph and factorization will change. If the window needs to be changed, edit the last part of the interact to obtain a smaller or larger window as needed. Click back into the interact code, update just that one line, and then click on **evaluate** underneath the cell.

Students can explore such challenges as finding two quadratics that cross the x -axis at, for example, $x = -2$ and $x = 4$. Two answers are $(x + 2) \cdot (x - 4)$ and

```
var('a')
var('b')
var('c')
@interact
def _quadratic(a=1,b=2,c=-3):
    print(factor(a*x^2+b*x+c))
    show(plot(a*x^2+b*x+c,(x,-10,10)))
```

(a)

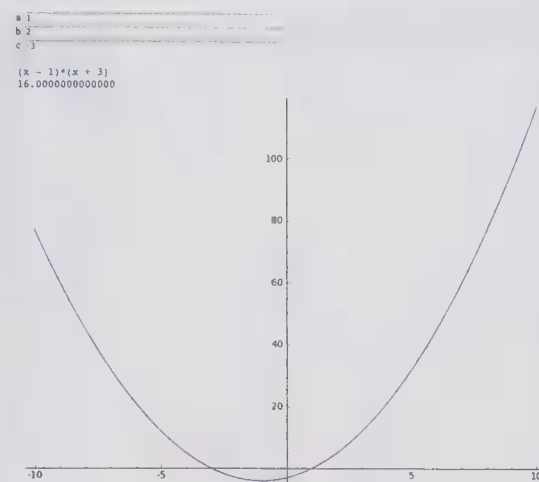


(b)

Fig. 9 The code in (a) sets up an interact to produce the graph and factoring in (b).

```
var('a')
var('b')
var('c')
@interact
def _quadratic(a=1,b=2,c=-3):
    print(factor(a*x^2+b*x+c));print b^2-4.0*a*c
    show(plot(a*x^2+b*x+c,(x,-10,10)))
```

(a)



(b)

Fig. 10 Once the code is adjusted (a), the discriminant of $f(x) = x^2 + 2x - 3$ is included in the interact output (b).

Cast Your Online Vote Today!

The **NCTM 2013 Board of Directors online election** is underway. Make sure to check your email for information about casting your vote, learning about the candidates, as well as nominating future candidates.

This election is exclusively online; voting instructions were sent to individual members who were current as of August 9, 2013 and had updated email addresses. If you have questions or need assistance, please contact Election Services Corporation at 1-866-720-4357 or email ctmhelp@electionservicescorp.com.

**All votes must be received by October 31st.
Be heard and cast your vote today!**



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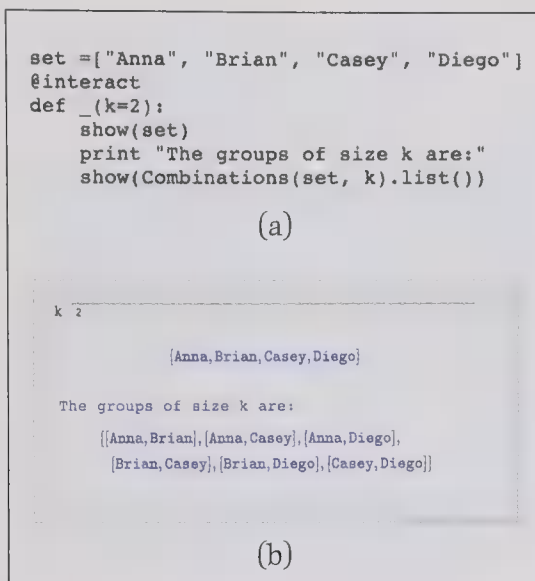


Fig. 11 This display (a) shows the code for an interact that finds groups of size n from a set; the output produced by this command appears in (b).

$-(x + 2) \cdot (x - 4)$, but students should be pushed to find more possible answers. As they play with Sage, they can discover that any constant times $(x + 2) \cdot (x - 4)$ yields the same result.

The discriminant can be incorporated into this interact by adding another **print** command after **print(factor(a*x^2+b*x+c))**. Click back into the code, insert a semicolon immediately after **print(factor(a*x^2+b*x+c))**, and type **print b^2-4.0*a*c** (see **fig. 10**); then click on **evaluate**.

By changing the values of the parameters, students can explore how to produce quadratics that stay above or below the x -axis. For example, $a = 2$, $b = 7$, $c = 10$ produces a quadratic with a discriminant of -31 and the graph of this quadratic will stay above the x -axis.

INTERACT 3: COMBINATIONS

The third interact quickly computes all possible combinations of the set specified before the interact, allowing students to address such natural questions as, How many groups of k students can be formed from a class of size n ?

We will first illustrate how to create groups of size two from a class of four students—Anna, Brian, Casey, and Diego. Type **set=["Anna","Brian","Casey","Diego"]** and then hit **return** without evaluating the cell. Next, type **@interact** and then press **return**. Directly underneath **@interact** type **def_(k=2):**, hit **return**, and type **show(set)**. Press **return** and type **print**

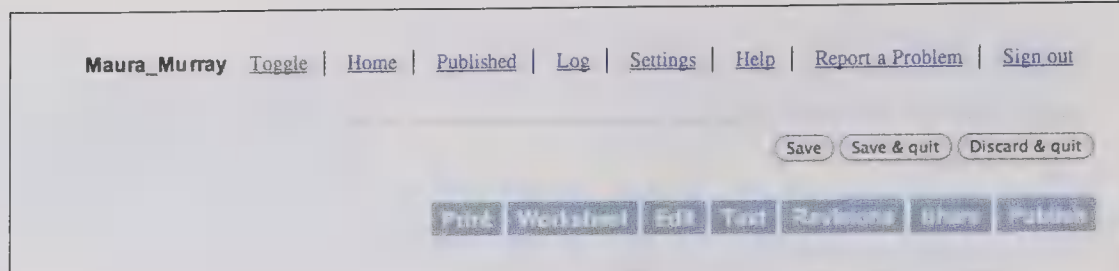


Fig. 12 This screen shot of the upper-right corner shows the **Share** command.

"The groups of size k are: ". Finally, press **return** and type **show(Combinations(set,k).list())**. See **figure 11**. Click on **evaluate** beneath the cell, and the following output shown in **figure 11b** will be displayed.

Students can explore conjectures about other group sizes by clicking in the input box and replacing the 2 with 0, 1, 3 or 4. After a change in the input box, hitting **return** makes the change.

Once the original interact (interact 3) is created, students can investigate other questions by altering the set to include their entire class or other groups that they are interested in pairing, tripling, or quadrupling off with.

A helpful pedagogical feature of Sage is the **Share** option, featured in the upper-right corner of the screen (see **fig. 12**). **Share** can be used to share your work with specific users. Students can set up free accounts, and then teachers can share templates that they would like their class to have by entering the students' usernames after clicking on **Share**. Interacts or just sample code can be prepared before class and shared; then, when students open their accounts, the shared file is available for use at home or in the classroom.

We have touched on only a small fraction of Sage's capabilities and pedagogical uses. A Sage tour, providing an overview of its many other capabilities, is available at http://www.sagemath.org/doc/a_tour_of_sage/. Quick reference cards are available at <http://wiki.sagemath.org/quickref>, with lists of basic (and not so basic) Sage commands. With a minimal investment—only time—students can use this dynamic computer algebra system to get a taste of computer programming and instantly share their results with their classmates.

ACKNOWLEDGMENT

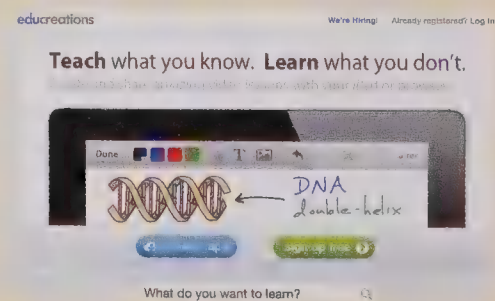
The author wishes to thank her instructors at the MAA PREP Workshop Sage: Using Open-Source Mathematics Software with Undergraduates (funding provided by NSF DUE 0817071) for all their Sage insights.



MAURA MURRAY,
mmurray@salemstate.edu,
teaches mathematics at
Salem State University in
Massachusetts. She is interested in mathematics teacher education, especially the integration of computer algebra systems into preservice teachers' experiences.

SURFING NOTE

More and more mathematics teachers are using digital content to help them find new and engaging ways to shift the way class time is being used. A website that provides access to online tools that allow teachers or students to create digital content through a "whiteboard" interface is www.educrations.com. After registering for free access to the site, teachers can record lesson materials that can be saved and then accessed directly by their students. The site also posts a wide range of content made available by other teachers. As with any online posting of such material, teachers need to review the content carefully before suggesting it as a resource for students. Educations also has a free tablet app. Teachers can use the convenience of the tablet interface to create, record, and upload their videos.



Mathematizing the World: An Invitation to Modeling

Modeling can mean many things. It is the mathematical representation of real-world behavior, but it also describes relationships among experiment, data, form, and function. What does mathematical modeling mean to you? How does modeling evolve in your classroom? What kinds of tasks invite students into the world of mathematical sense making?

The Editorial Panel of *Mathematics Teacher* invites teachers to share their experiences with modeling. We especially encourage submissions that help *MT* readers wrestle with complexities of mathematical modeling in classrooms. How do we help students discover a dynamic relationship between physical phenomena and mathematical structure?

Here are some prompts to get you started thinking and writing:

- How does your classroom culture promote modeling?

- How does the process of modeling unfold from initial assumptions to final conclusions?
- How do students proceed as they refine their models?
- How do you or your students determine whether a model is successful?
- How can assessment of a model be reconciled with assessment of student learning?
- How does our interest in modeling span the grades?

You may submit your completed manuscript for review by accessing mt.msubmit.net. Indicate that the manuscript is being submitted in response to the call Mathematical Modeling. Be sure to enter the call's title in the Department/Calls field. No author identification should appear in the text of the manuscript. For additional guidelines for preparation of manuscripts, see www.nctm.org/publications/content.aspx?id=22602.



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CALL FOR MANUSCRIPTS

Transforming Shapes Physically and Analytically

Constructing formulas “from scratch” for calculating geometric measurements of shapes—for example, the area of a triangle—involves reasoning deductively and drawing connections between different methods (Usnick, Lamphere, and Bright 1992). Visual and manipulative models also play a role in helping students understand the underlying mathematics implicit in measurement and make sense of the numbers and operations in formulas.

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via <http://mt.msubmit.net>. For more background information on the department and guidelines for submitting a manuscript, visit <http://www.nctm.org/publications/content.aspx?id=10440#delving>.

Edited by **Maurice Burke**, Maurice.Burke@utsa.edu
University of Texas at San Antonio

J. Kevin Colligan, jkcolligan@verizon.net
RABA Center of SRA International, Columbia, MD

Maria Fung, mfung@worchester.edu
Worcester State College, Worcester, MA

Jeffrey J. Wanko, wankojj@muohio.edu
Miami University, Oxford, OH

The authors—two mathematicians and one mathematics educator—wondered about physical models: What assumptions are made in using representations to help students, teachers, and mathematicians better understand mathematical ideas? When should we be cautious or skeptical about what physical or visual models suggest? These questions led to discussions within our department that involved ideas of proof and the extent to which visual models can convince others of the validity of mathematical claims. In addition, we began to consider the mathematical tools used to generalize measurement methods. Here we focus on understanding area formulas for circles and ellipses, but readers are encouraged to think more broadly about the benefits and limitations of using physical representations as tools to understand abstract ideas.

NCTM's Measurement Standard recommends that students be given opportunities to approximate and apply informal ideas of upper and lower bounds in measurement situations (2000, pp. 44–47). For example, in developing an understanding of the area enclosed by a circle, students might inscribe a square with diagonal lengths of $2r$ (area $2r^2$) and circumscribe a square with side lengths of $2r$ (area $4r^2$), the first steps used by Archimedes to approximate the area inside a circle (Özgün-Koca 2007). Using this model, we find that the area

inside the circle is between $2r^2$ and $4r^2$. Averaging those measurements produces a closer approximation to the formula for the area of a circle as $3r^2$.

A second way to make sense of the area inside a circle affords opportunities to develop ideas of successive approximations and the limit, concepts foundational to measurement and calculus. A paper disk of radius r is cut into wedges and rearranged (see **fig. 1**) into a “bumpy parallelogram” (Bennett, Burton, and Nelson 2010). Successive sectioning produces a closer and closer approximation to a parallelogram with $h = r$ and $b = \pi r$. Note that we are defining π as the value of the ratio of the circumference to the diameter. Multiplying the base by the height of this “parallelogram” again suggests the formula for the area of the circle— πr^2 .

A third model for understanding the development of the formula for the area inside a circle is found in Japan’s national mathematics curriculum (Hironaka and Sugiyama 2006). This model uses concentric rings to fill the disk. The textbook describes rings made of rope that, when straightened and stacked, form a “triangle” (see **fig. 2**). We discovered that waxed string (commercially called Bendaroos®) provided a promising physical material for modeling the actions, but we became interested in whether the method itself was mathematically valid. Analysis of this circle-to-triangle method is the focus of this article.

The physical model reveals that each string is longer than the previous one by a constant amount. That makes sense because the strings correspond to circumferences of circles with radii that are a constant amount (width of one string) larger with each circle. Moreover, each circumference is directly proportional to its radius, and 2π is the constant of proportionality. If we imagine infinitely many infinitely thin strings, the shape produced is a triangle with base $2\pi r$ and height r (or with base r and height $2\pi r$), as suggested in **figure 2**. The area of the circle can be found using the formula for the area of a triangle ($A = bh/2$); in these cases, the area is $(2\pi r) \cdot r/2 = r \cdot (2\pi r)/2$, which again simplifies to πr^2 . This model could be used to motivate the use of the integral to find the area of a circle:

$$\int_0^r 2\pi x \, dx = \frac{2\pi r^2}{2} = \pi r^2$$

The model constructed with Bendaroos seemed to be a useful tool to support understanding of the area of a disk. However, we wondered about the validity of “uncurving” the concentric strings that filled the disk into straight strings that filled the triangle. Our concern was that the “cut” to the center of the disk (the red line segment in the first image

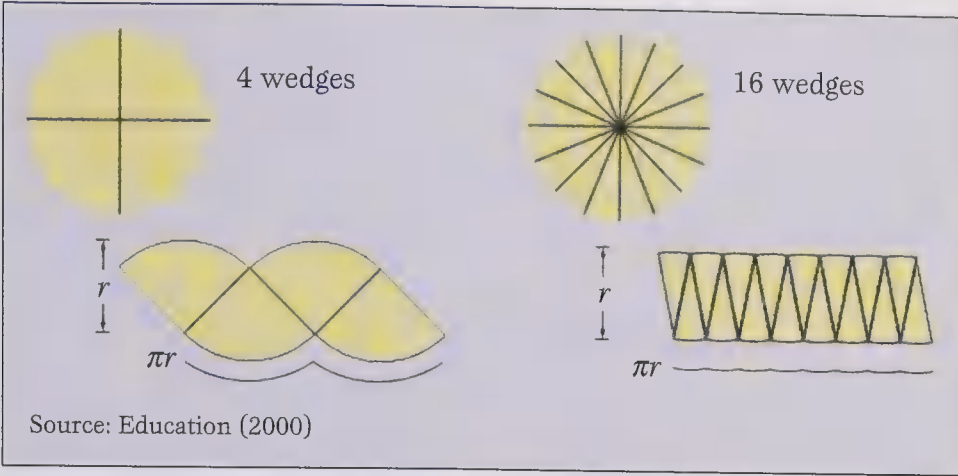


Fig. 1 The area of a circle can be understood by rearranging wedges into a “bumpy parallelogram.”

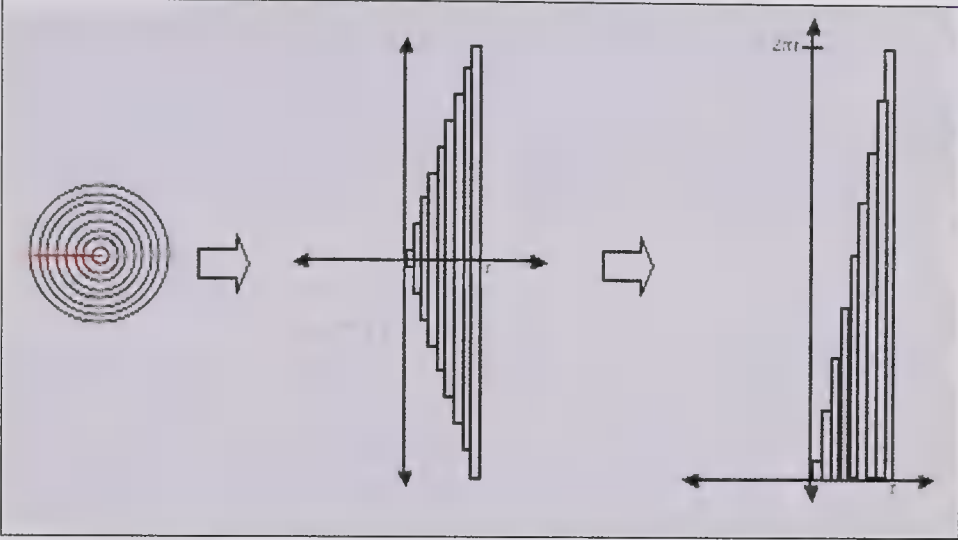


Fig. 2 The area enclosed by a circle can be understood using concentric rings that are cut along the red segment, unwrapped, and arranged to make two triangles.

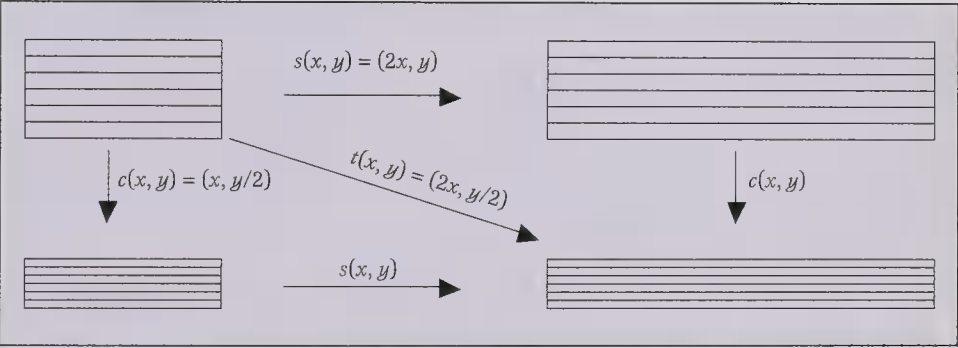


Fig. 3 Functions that stretch and compress can be explored.

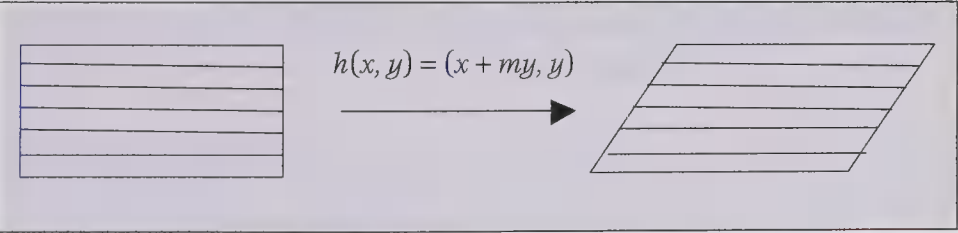


Fig. 4 One example of an area-preserving function is the shearing function.

in **fig. 2**) had had a length of r , but straightening the strings appeared to lengthen the “cut length” (sides of the triangle). This seemed to present a situation in which a physical representation of an abstract model may introduce anomalies. Certainly lengths (if not areas) were being distorted. Further,

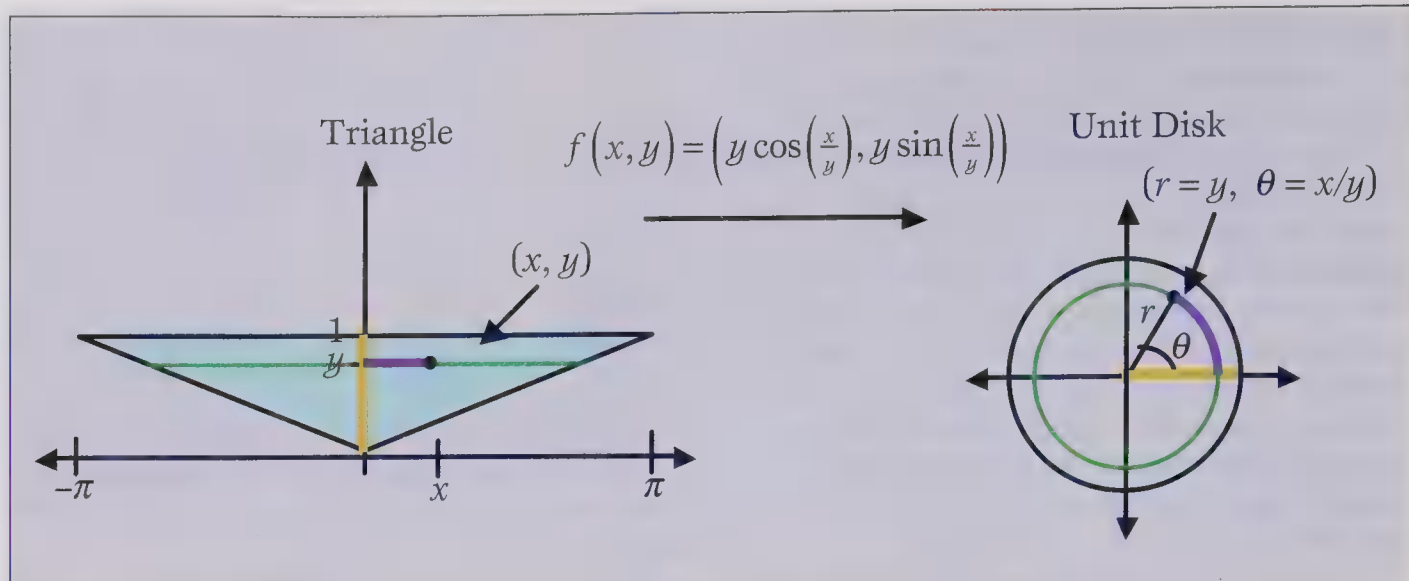


Fig. 5 The process of transforming the triangular region into the circular disk involves curving the segments.

this may be a case in which physical materials are underscoring problems with the theoretical model, problems that might be investigated using mathematical tools.

A TRANSFORMATION FUNCTION FROM TRIANGLE TO CIRCLE

To consider the kinds of distortions that might be occurring in the circle-to-triangle transformation, we examined some simple transformations that distort shapes. Imagine a rectangular space filled by thin strips (see **fig. 3**). Stretching those strips in the horizontal direction, using the function $s(x, y) = (2x, y)$, clearly changes the area of the rectangle. But a stretch in one direction could be compensated for by a compression in another, $c(x, y) = (x, y/2)$. We might model these two actions with rubber strips that are stretched to twice their original length and “narrowed” to half their original width. The composition of both functions produces $t(x, y) = (2x, y/2)$, a function that distorts shapes but preserves area.

A second example in which distortion takes place is a shearing function (see **fig. 4**). Although the horizontal segments slide against one another, the height remains unchanged, and the total area is preserved. Functions such as $t(x, y)$ and $h(x, y)$ demonstrate that area can remain constant even when lengths are distorted.

To determine whether the transformation that straightens concentric rings preserves area, we tried to develop a function to map points inside the disk to corresponding points inside the triangle. We found that it was easier to reverse the process, starting with the triangle of straight segments and curving those segments into concentric circles (see **fig. 5**).

We imagined a triangle with height 1 filled with horizontal line segments (pieces of waxed string), one of which is shown in green (see **fig. 5**). Then we considered a point (x, y) on that line segment

and built a function that maps (x, y) to a point within the unit disk on a green circle of radius y , centered at the origin and with circumference the length of the green line segment. We mapped the midpoint of the line segment to the positive x -axis of the unit disk, curving the positive x -direction of the triangle’s segment counterclockwise from the horizontal axis.

The point (x, y) from the triangle maps to the polar coordinates (r, θ) on the circle where $r = y$ and $\theta = x/y$. The radius of the green circle is y . The meaning of θ is tricky. We reasoned that the distance around the purple arc of the circle is x (from the diagram of the triangle) and 1 radian (the angle made by 1 radius length) is given by an arc length of y . So, $\theta = x/y$ radians (i.e., a fraction of 1 radian). Changing the coordinates for this point from polar to Cartesian coordinates, we see that (x, y) maps to

$$\left(y \cos\left(\frac{x}{y}\right), y \sin\left(\frac{x}{y}\right) \right).$$

Expressed as a function, we have

$$f(x, y) = \left(y \cos\left(\frac{x}{y}\right), y \sin\left(\frac{x}{y}\right) \right);$$

$$-\pi y \leq x \leq \pi y; 0 < y < 1. \quad (1)$$

This function maps a point (x, y) inside the triangle to a point on the unit disk.

DOES THIS TRIANGLE-TO-CIRCLE TRANSFORMATION PRESERVE AREA?

As shown in **figure 3**, transformation functions can impose distortions in the resulting image that can change area. We used a linear algebra tool, the determinant of a transformation matrix, to investigate whether an area change occurs with our triangle-to-circle function. Our discussions then focused on the questions, What is the determinant, and how does it detect size changes? Recall

that a 2×2 matrix of numbers can be thought of as representing a linear function from the plane to itself. We do this by representing a point (x, y) in the plane as a column vector and multiplying matrix M by the vector to give a new column vector, representing the image of this point under the linear function.

As an example, we will use the matrix

$$M = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

multiplied by

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

to give

$$\begin{pmatrix} 3x + y \\ x + 2y \end{pmatrix} = (3x + y, x + 2y).$$

The “linearity” of the map is seen by the fact that the images of the straight-line vectors—the sides of the unit square in **figure 6**,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

—remain the straight-line vectors

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

the sides of the parallelogram in **figure 8** in the transformation. In **figure 6**, the image of the unit square with the origin as its lower-left corner and area = 1 is a parallelogram spanned by the vectors formed by the columns of the matrix M . The area of this parallelogram is 5 square units and can be measured directly on the grid or computed in terms of the vectors as the absolute value of the determinant of M : $3 \cdot 2 - 1 \cdot 1 = 5$. The function represented by M distorts both sides of the unit square and, in so doing, also scales the area by a factor of 5.

Why is the determinant of the transforming matrix also the factor by which the area is scaled? This question took our inquiry discussion group on a trigonometric adventure that is nicely summarized by Konvisser (1981).

Under a linear transformation defined by a matrix M , each unit square becomes a parallelogram with sides defined by the matrix column vectors. The number of parallelograms in the geometric image is the same as the number of squares in the original (see **fig. 6**). Because each square is scaled by $\det(M)$ to produce each parallelogram, the

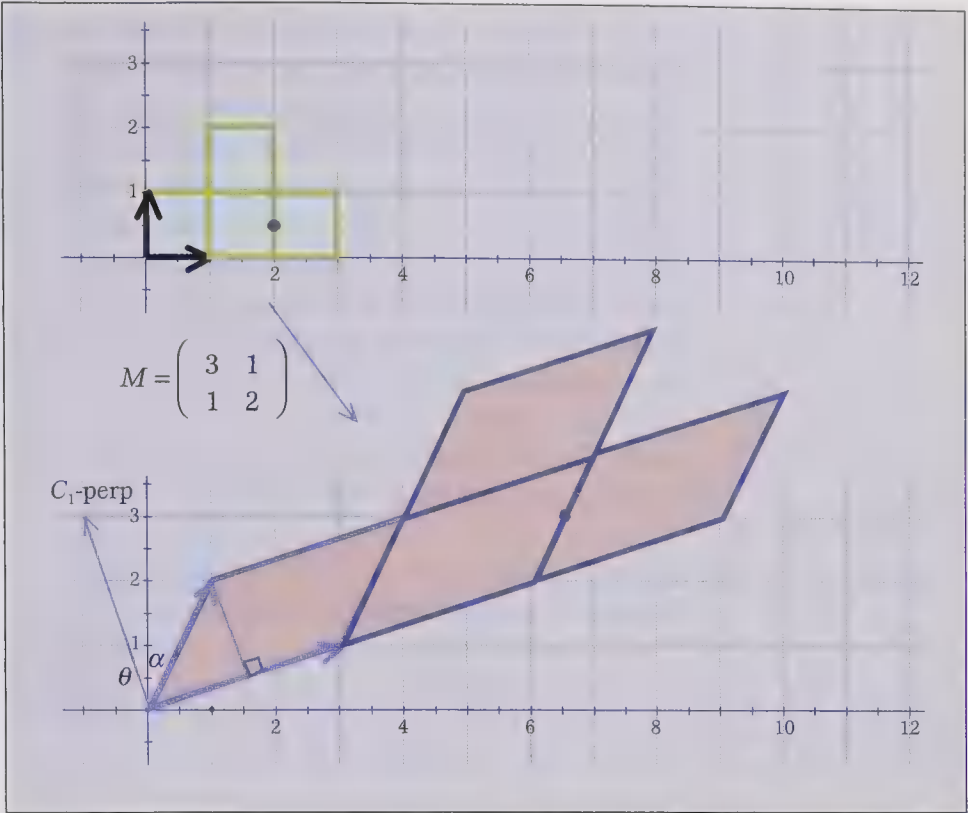


Fig. 6 A matrix transforms squares to parallelograms and rescales area by the determinant.

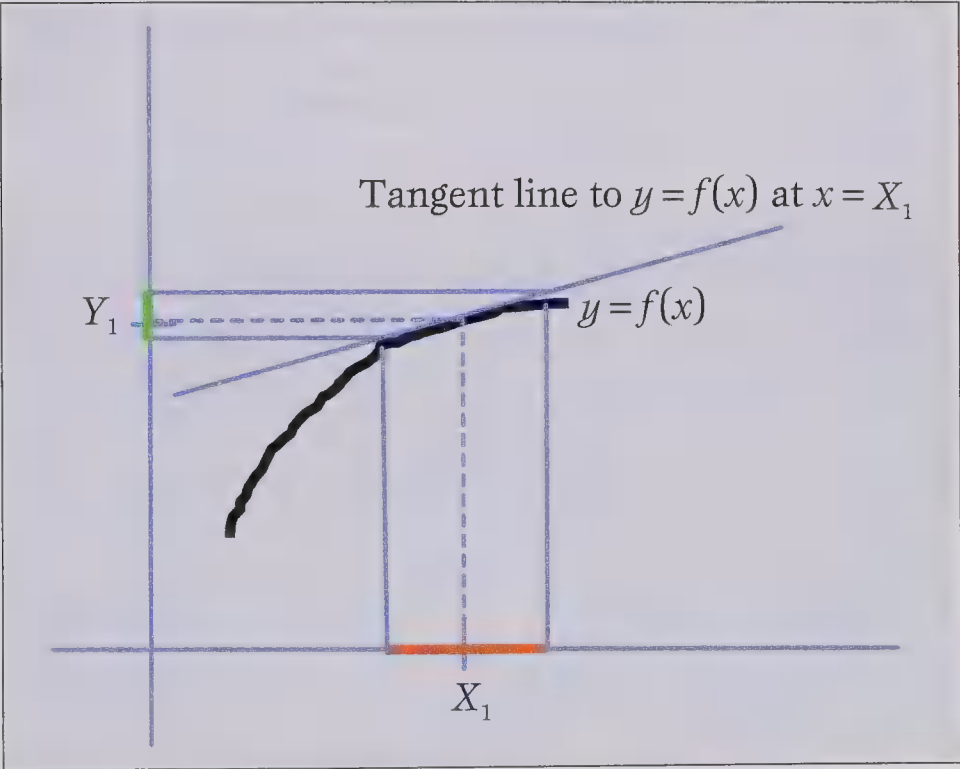


Fig. 7 A nonlinear function does not generally preserve length.

entire region is scaled by a factor $\det(M)$.

The determinant works well to find the scale change of linear functions of two variables, but our triangle-to-circle function is nonlinear. To investigate whether any scaling of area occurs with our transformation, we needed to “linearize” the function. We do so by using a matrix of partial derivatives, called the *Jacobian matrix*.

To help us understand why the Jacobian matrix was relevant, we considered a simpler case, a one-variable situation. We can think of a

nonlinear function of one variable, $f(x)$, as mapping x -values (such as X_1) to corresponding y -values (Y_1) and intervals on the x -axis to corresponding intervals on the y -axis (see **fig. 7**). The interval on the x -axis (orange segment) can be mapped to an approximate interval on the y -axis (green segment). Because the function $f(x)$ is nonlinear, we use the tangent line as an approximation. Note that $f(x)$ is not length preserving (i.e., the green segment is not the same length as the orange segment) for values of x near X_1 . In fact, a function will be length preserving only when the slope of the tangent line, $f'(x)$, is equal to ± 1 (i.e., when the linear function $f(x) = \pm x + b$).

For functions of two variables, we will use the Jacobian matrix of partial derivatives,

$$Df(x, y) = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix},$$

as an analog to the derivative of a single-variable function. By evaluating the Jacobian matrix at a particular point (a, b) , we find that the resulting numerical matrix describes a linear approximation of the function around that point. So if R is some small region located around (a, b) , then the area of the image $f(R)$ is approximately the area of R times the absolute value of the determinant of the Jacobian matrix at (a, b) , and this approximation improves as we choose the region R to be smaller and smaller. Just as $f'(x) = \pm 1$ indicates a length-preserving function, when the determinant of the Jacobian of $f(x, y)$ at a particular point is ± 1 , the area of a small region around that point is preserved when f is applied to it.

Further, we can show that, if $\det(\text{Jacobian})$ is precisely 1 (or -1) at every point, then applying the function f to a region will give an image with exactly the same area. By verifying that this triangle-to-circle function, equation (1), has a Jacobian matrix with determinant of ± 1 at every point, we rigorously show that $f(x, y)$ relates two figures (triangle and circle) of the same area.

As an example of how the Jacobian matrix detects distortions in area, the composite transformation $t(x, y) = (2x, y/2)$ introduced in **figure 3**, can be shown to be area preserving. In this case, the partial derivatives are constants and the Jacobian matrix,

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

has a determinant of 1. With this triangle-to-circle function, the Jacobian matrix is as follows:

$$Df(x, y) = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix} = \begin{pmatrix} -\sin\left(\frac{x}{y}\right) & \cos\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)\sin\left(\frac{x}{y}\right) \\ \cos\left(\frac{x}{y}\right) & \sin\left(\frac{x}{y}\right) - \left(\frac{x}{y}\right)\cos\left(\frac{x}{y}\right) \end{pmatrix}$$

Computing the determinant of $Df(x, y)$ and simplifying, we see that it is -1 for (x, y) in the triangle's interior and $y \neq 0$:

$$\begin{aligned} \det(Df(x, y)) &= -\sin^2\left(\frac{x}{y}\right) + \left(\frac{x}{y}\right)\sin\left(\frac{x}{y}\right)\cos\left(\frac{x}{y}\right) \\ &\quad - \cos^2\left(\frac{x}{y}\right) - \left(\frac{x}{y}\right)\sin\left(\frac{x}{y}\right)\cos\left(\frac{x}{y}\right) \\ &= -\left(\sin^2\left(\frac{x}{y}\right) + \cos^2\left(\frac{x}{y}\right)\right) = -1 \end{aligned}$$

We are thus assured that the function mapping all points (x, y) (within the triangle) to points on concentric circles in the interior of the unit disk did not change the area of the image—that is, $f(x, y)$ is area preserving. So if there were distortion in one direction, it was compensated for by another distortion, and the area of the triangle was ultimately preserved in the disk.

MATHEMATICAL AND PEDAGOGICAL INSIGHTS

Where has this investigation taken us? Our initial questions seemed fairly innocent: Can physical materials such as Bendaroos be used to understand measurement formulas better? If so, what are the limitations of the materials, and how can more abstract mathematical models help overcome some of these limitations? As we engaged in a collaborative inquiry into these questions, one that required higher-level mathematics, we noticed that our group was also using mathematical processes and habits of mind that we encourage students to practice—reasoning about mathematical phenomena, looking for counter-examples, examining each part of an argument, exploring why analytical tools work, and supporting one another in clarifying ideas. We discussed levels of proof and proving: What does it take to convince ourselves and others of the validity of assertions? (Dodge, Goto, and Mallinson 1998; Sowder and Harel 1998).

This process also helped us think about uses of physical models more generally in mathematics education (Arcavi 2003). Physical materials are rough approximations for abstract ideas. Sometimes they suggest real-world referents for abstract mathematical concepts or provide a handle on solving a problem. At other times, the model or metaphor might lead to overstated or incorrect conclusions. So it is important for mathematicians as well as teachers and learners of mathematics to consider their selection of physical

models when representing abstract ideas in particular cases. Our group is now exploring under which geometric conditions the concentric strings model can be used to determine area.

Finally, this exploration increased our appreciation of collaboration. The cross-disciplinary interests in our conversations energized our group as we integrated ideas from mathematics and mathematics education.

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MARGARET RATHOUZ, rathouz@umich.edu, an elementary school mathematics educator, and **JOHN CLIFFORD**, jcliff@umich.edu, a secondary school mathematics educator, are associate professors in the Department of Mathematics and Statistics at the University of Michigan-Dearborn. **CHRISTOPHER NOVAK**, cnovakmath@gmail.com, is a software engineer at GREE International. All three authors are interested in discussing and making sense of the mathematics they teach and use.

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- Aim for an informal style that appeals to *MT* readers (review recent issues of the journal for examples).
- Manuscripts that feature the use of technology to teach and learn mathematics should be as generic as is reasonable in describing products.
- Proofread your manuscript in advance for English usage, mathematical correctness, and accuracy of references. Use a spell checker before submitting your manuscript and recheck it for grammar, completeness, spelling of all proper names, and full identification of all organizations and groups named by initials or acronyms.

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1. Double-space all material, including quoted matter, lists, tables, notes, references, and bibliographies. Manuscripts should not exceed 2500 words (ten pages), not including tables and figures. Include only tables and figures essential to understanding.
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Common Core Mathematics in a PLC at Work™: Leader's Guide, Timothy D. Kanold and Matthew R. Larson, 2012. Grades Pre-K-12, 155 pp., \$24.95 paper. ISBN 978-1-936765-47-8. Stock no. 14387. National Council of Teachers of Mathematics and Solution Tree Press; www.nctm.org.



Professional Learning Communities (PLCs) focused on the understanding and implementation of the Common Core State Standards for Mathematics (CCSSM) is, the authors argue, one viable response to the literature that demonstrates the challenges created when teachers work in isolation. Kanold and Larson further argue that PLCs are needed to bring about sustainable change in schools in lieu of what they call “one-shot” professional development. To support the work of PLCs and their leaders, this book provides a wealth of resources to address critical areas of need as well as paint a picture of the promise of a PCL when all entities are truly collaborating.

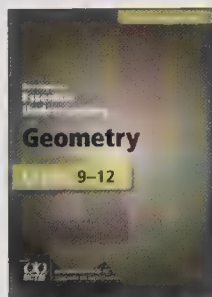
The book's strength is its practical examples. Incremental actions in the form of steps, stages, or tiers are suggested to

help collaborative groups begin and sustain their work while focused on student learning, development, and assessment. The examples provided allow leaders and teachers to “see” what each step, stage, or tier might look like. Tools and rubrics are embedded throughout the text and help illuminate major ideas in each chapter.

The questions at the end of chapters along with the variety of included tools make this book a helpful resource for schools interested in developing PCLs, whether their focus is on the Common Core or other important issues. In addition, the tools and rubrics are available online and can be modified by individual groups or leaders to fit their specific needs.

—Stacy Reeder
University of Oklahoma
Norman, OK

Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 9-12, Nathalie Sinclair, David Pimm, and Melanie Skelin; Rose Mary Zbiek, series ed., 2012. 109 pp., \$35.95 paper. ISBN 978-0-87353-692-9. Stock no. 14123. National Council of Teachers of Mathematics; www.nctm.org.



This recent addition to NCTM's Essential Understanding series highlights four key ideas that geometry teachers need to consider in the teaching of geometry: diagrams,

variance and invariance, definitions, and proof. The understanding and assessment of these ideas, along with the vertical and horizontal alignment of geometry topics, is the book's focus.

One key area that the authors emphasize is the importance of variance—that is, change—in the study of geometry. One highlighted example is the intersection of a circle with a line. Specifically, if a chord is rotated at a point of intersection with the circle so that it becomes a

tangent, then that line is also perpendicular to a radius at that point. Further, this explanation verifies the root meaning of the word *right*, which relates to balance (p. 12). In other words, the tangent is balancing on the point of tangency; if not, it would be intersecting the circle at two points. Word origins and roots are used throughout the book to help clarify and illuminate many geometric terms.

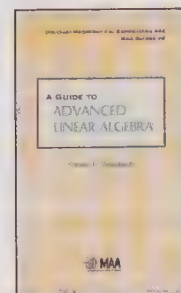
While reviewing this book, I had many “aha!” moments when the authors made connections between geometry and other areas of mathematics. For example, when decomposing a triangle into a rectangle to develop the area formula, students need to know the algebra concepts of slope and midpoint of the sides of a triangle (p. 54).

Reading this book will expand and strengthen any experienced geometry teacher's knowledge. Note: Thinking is required!

—Lynn Mitzel
Fargo Public Schools
Fargo, ND

FROM OTHER PUBLISHERS

A Guide to Advanced Linear Algebra, Stephen H. Weintraub, 2011. 255 pp., \$39.95 cloth. ISBN 978-0-88385-351-1. The Mathematical Association of America; www.maa.org.



A Guide to Advanced Linear Algebra is the newest addition to the Dolciani Mathematics Exposition Series (no. 44) and the MAA guides (no. 6). Its approach to linear algebra is strictly theoretical; no

computational issues are addressed. The book assumes that readers know most of the basics in linear algebra; thus, it is certainly not suited for beginning students.

In addition, this book is not suitable for students with a rich knowledge of algebra, who would already be familiar

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with the material presented here. Those who would benefit from such a book are upper-division undergraduate students or beginning graduate students who lack formal depth in proof of theorems of linear algebra as well as teachers who need a refresher in these topics.

The book is well organized. It approaches linear algebra from a purely algebraic perspective—namely, the concepts of vector spaces and linear transformations. Definitions, remarks, lemmas, examples, theorems, and proofs are labeled simply and, most of the time, clearly. The author's intent is evident in his organization of the text. He starts with basic definitions and theorems of vector spaces and linear transformations and then moves to matrices of linear transformations; determinants, eigenvalues, and eigenvectors; and then inner product spaces. The beginning synopsis of each chapter gives a view of the author's intent for that chapter and allows readers to focus on the content.

This book provides readers with valuable knowledge. It would be most useful as a reference book for teachers who wish to revisit the major ideas and proofs of the topics of linear algebra.

—Cynthia Barb
Kent State University at Stark
North Canton, OH

“Heterogenius” Classrooms: Detracking Math and Science—A Look at Groupwork in Action, Maika Watanabe, 2012. Foreword by Michelle Fine. 98 pp. + DVD, \$36.95 cloth. ISBN 978-0-8077-5246-3. Teachers College Press; www.teacherscollegepress.com.



In “*Heterogenius*” *Classrooms*, Watanabe presents a framework for instruction in a detracked, heterogeneous classroom. She defines *detracking* as “universally accelerating all students.” Through her research at three urban charter schools, she found that all students make academic achievement gains in a detracked program. The book and DVD explain the benefits of detracking and how to instruct effectively through group work. The book provides four lesson

plans, rubrics, handouts, and reflections that model group work. Watanabe also addresses the necessary additional supports for students and teachers for all to be successful through this framework.

“*Heterogenius*” *Classrooms* is intended for use by all educators, from teacher education programs to school- and district-level administrators to classroom teachers. The book and DVD can be used to spark discussion and inspire ideas about how to implement effective group work. Although the setting for this framework is urban charter schools, the framework can be applied to schools of all sizes and demographics. The book and DVD highlight the six characteristics of group-worthy tasks and provide four examples of such tasks. I wish the DVD had included a lengthier clip of a group-worthy task in action and had featured the teacher's role in facilitating the learning through the task.

Although detracking may not be a reality for all schools, Watanabe's ideas are worth considering as educators strive to provide effective instruction for all students.

—Fenecia Foster
Sawyer Public School
Sawyer, ND

The Hidden Mathematics of Sport, Rob Eastaway and John Haigh, 2011. 199 pp., \$15.52 (U.S) cloth. ISBN 9781907554223. Anova BookStore; www.anovabooks.com.



This book is fun! The fact that most of the sports mentioned here are uncommon ones for a U.S. audience—not ignored, just underappreciated—puts the reader in a novel framework yet with an old friend: mathematics.

The fact that sport and mathematics are intertwined is not news to anyone, particularly to any baseball, basketball, or hockey fan, but this relationship exists in many other sports as well. Certainly, we are all aware of tennis and soccer (make that “football”!), but don't forget about rugby, cricket, snooker, darts, horse racing, high jumping, triathlons, and all the Olympic sports, just to name a few. If this list sounds Anglocen-

tric, you've divined the book's origin.

Don't take up this book expecting to be “schooled” in sports-generated mathematical nuances or (for the most part) references to advanced mathematics. But the mathematics here is approachable and intelligible at the secondary school level—counting arguments, game theory, some very simple physics, tournament design, and elementary statistics. Even the efficacy of cheating is addressed. And the authors have added a few teaser problems to each chapter to make you think about both the sport itself and the mathematics lurking behind it. They also provide ample additional references to pursue suggested topics more fully.

I won't spoil your read, but here are a few intriguing questions discussed in the book:

- In shot put, are you better off adding 1 percent to the height of the release or 1 percent to the speed?
- Who or what were Running Rein and Flocton Grey?
- What's a googly?
- How does Pythagoras's theorem apply to the triathlon?
- Statistically, what is the best tactic for running an 800 meter race?
- What's unusual about the football road signs in Britain?

You may collect some ideas for classroom projects, you will certainly enjoy the writing, and you will come away pleased, a little challenged, and not drained or pushed over the edge mathematically. Definitely worth reading!

—J. Kevin Colligan
RABA Center, SRA International
Columbia, MD

Who's #1? The Science of Rating and Ranking, Amy N. Langville and Carl D. Meyer, 2012. 266 pp., \$29.95 cloth. ISBN 978-0-691-15422-0. Princeton University Press; www.press.princeton.edu.



The authors' stated purpose is to provide a single resource that describes several techniques and uses of both ratings and rankings, including some new

strategies that they have devised themselves. The audience they identify is vast, from mathematicians to sports enthusiasts to gamblers. Although they suggest that some fundamental knowledge of linear algebra and optimization is helpful, they encourage those without this knowledge to read the book anyway. However, some examples are not helpful for those without the mathematical grounding; at other times, the mathematics is overwhelmed by the number of data points in the example.

The book contains some fascinating applications, such as the 2000 U.S. presidential election, but focuses more on sports examples, some of which become tedious. The authors also provide a reference list that offers numerous avenues

for pursuing details and nuances.

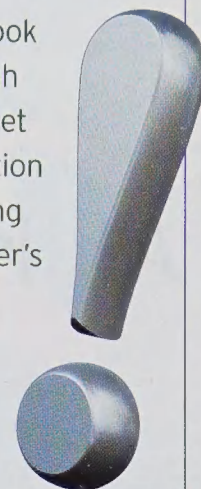
In several instances, the authors begin with a theory and then illustrate the theory with an application; in others, they build to the theory from an application. This inconsistency, which makes some chapters appealing to the nonmathematical reader and other chapters appealing to the mathematical reader, can make the text difficult to follow at times.

Overall, the book does achieve the authors' stated purpose. Readers will find many interesting ideas as they grapple with the complexities of the science of rating and ranking.

—Bob Horton
Clemson University
Clemson, SC

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New from NCTM: The Essential Guide to Navigating Your First Years of Teaching Secondary Mathematics

THE FUTURE. INSPIRING TEACHERS. ENGAGING STUDENTS. BUILDING THE FUTURE. INSPIRING

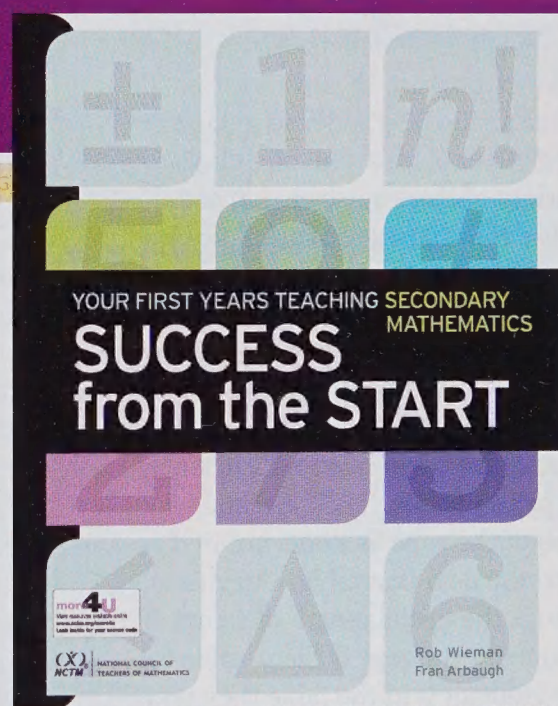
Success from the Start: Your First Years Teaching Secondary Mathematics

BY ROB WIEMAN AND FRAN ARBAUGH

You just signed your first contract to teach secondary math. You're excited but you have many questions and concerns:

- What do I do when students don't "get" the lesson?
- What about students who struggle with math they supposedly learned in elementary school?
- How do absent students make up the work?
- Do I assign seating or let students sit wherever they want?
- Should I let students work in groups?
- How much homework should I assign and grade?

Based on classroom observations and interviews with seasoned and beginning teachers, *Success from the Start: Your First Years Teaching Secondary Mathematics* offers valuable suggestions to improve your teaching and your students' opportunities to learn. The authors explore both the visible and invisible aspects of teaching and offer proven strategies to make the work meaningful—not merely manageable. Success from the start means being prepared from the start. This book not only teaches you how to be an effective math teacher but also gives you the tools to do it well.



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MY FAVORITE
lesson

Jamie-Marie L. Wilder
and Molly H. Fisher

All Knotted Up

Our favorite lesson is a hands-on activity that helps students visually “tie” (pun intended) the concepts of rate of change and y -intercept together in a meaningful context using strings and ropes. Students tie knots in ropes of various thicknesses and then measure the length of the rope as the number of knots increases. We provide clothesline, twine, bungee cord, and other ropes found at local crafts, sporting goods, and home stores. We avoid very thin string, such as thread or knitting yarn, because the knots are small and the string length does not change enough to explore a rate of change. A variety of thicknesses is important because this allows for variability in the rates of change.

Students complete the activity in pairs, with each pair choosing a type of string and cutting off a length of at least 30 cm. Students will be asked to tie ten knots, so they will need a sufficiently long piece of rope. They measure the length of the rope as knots are tied and record the data in a table. They first measure the initial length of string. This initial measurement is the y -value for the x -value of 0 in the table they will

create. Students then tie the first knot and measure the length of the string, providing the y -value for the x -value of 1 in their table. They continue until they have tied ten knots in the string. In addition to using the data to discuss rates of change and y -intercept, we also discuss correlation, scatter plots, line of best fit, estimation, and linear regression.

We ask students to use the data in their tables to answer a series of questions, including these:

1. Describe what trends, if any, you see in the data.
2. Do the data represent a positive or a negative correlation?
3. Create a scatter plot and determine a line of best fit for your data. Then estimate the number of knots that must be tied so that your string is 12 cm long.
4. Choose two data points and calculate the linear equation.
5. In real-world terms, what does it mean for x to be zero? For y to be zero?
6. In real-world terms, what does “rate of change” mean in this problem?

When finished, students turn in their knotted strings for display during debriefing. They can further investigate rate of change and y -intercept with differing gauges and lengths of string and compare results with those of their classmates. This comparison generally sparks some interesting discussions; students are always curious to see how other teams tied their knots. They are

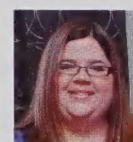
surprised to discover that successive knots can be tied at any interval and the results are the same. This activity can be reversed to model a positive correlation by issuing the strings already knotted and asking students to measure the length as they untie the knots.

We enjoy watching learning happen in this lesson as students see the connection between number of knots and rope length, creating a real-world context for rate of change and y -intercept. Students are able to collect their own data, in the classroom (without mess!), and analyze their own work. They also like to have contests to determine which types of rope will allow the most knots to be tied in the 30-cm length, further applying the rates of change that they discovered during the lesson.

This lesson is our favorite because students are so actively involved in their own mathematical learning.



JAMIE-MARIE L. WILDER, jamie.wilder@lincoln.kyschools.us, teaches eighth-grade mathematics at Lincoln County Middle School in Stanford, Kentucky. She is interested in algebraic thinking through contextual modeling. **MOLLY**



H. FISHER, molly.fisher@uky.edu, is an assistant professor of mathematics education in the STEM Education department at the University of Kentucky in Lexington. She works with preservice mathematics teachers and studies teacher stress, burn-out, and retention.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Edited by **Jennifer Wexler**, wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL

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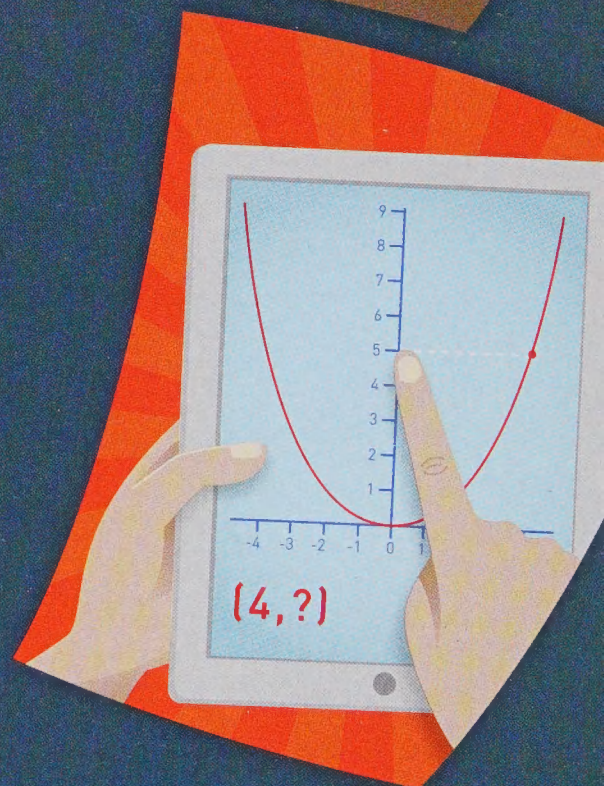


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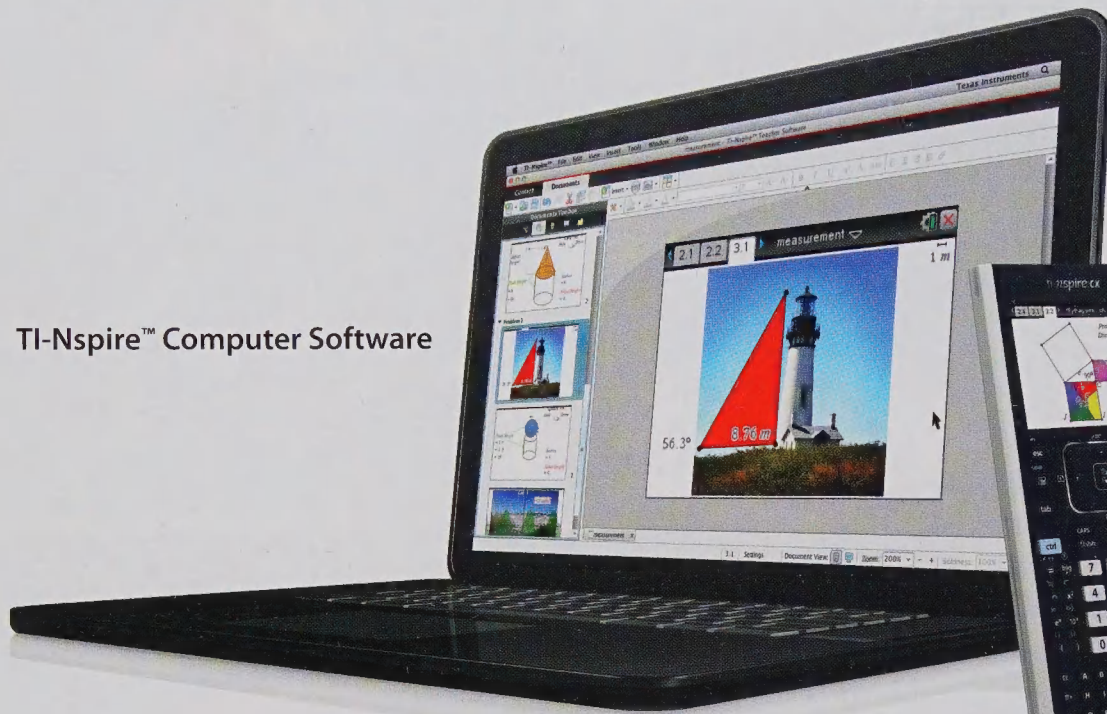
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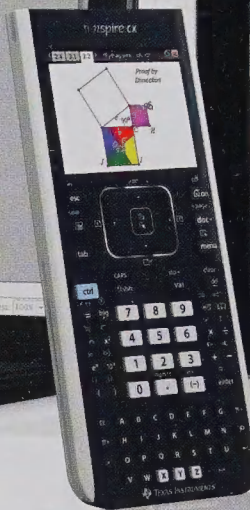
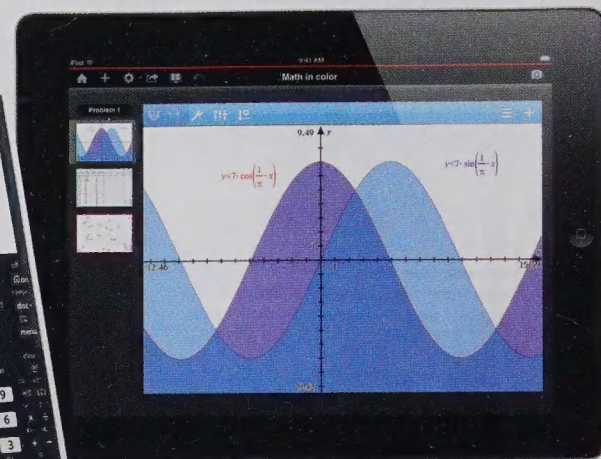
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